APPENDIX TO ESTIMATING WAITING TIMES WITH THE TIME-VARYING LITTLE'S LAW

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Abstract

When waiting times cannot be observed directly, Little's law can be applied to estimate the average waiting time by the average number in system divided by the average arrival rate, but that simple indirect estimator tends to be biased significantly when the arrival rates are time-varying and the service times are relatively long. This study shows that the bias in that indirect estimator can be estimated and reduced by applying the time-varying Little's law (TVLL). The new methods are shown to be effective in estimating the bias in the indirect estimator and reducing it, using simulations of multi-server queues and data from a call center. This appendix provides additional details about those experiments.

Keywords: Little's law; time-varying Little's law; $L = \lambda W$; estimation; estimation bias; estimating the average wait

Contents

1 Overview

We present supporting material in this appendix to the main paper. In §2 we present additional results for the simulation experiments discussed in Section 8. In §3 we present additional results for the call center data examined in Section 9. Throughout this appendix, we refer to equations as numbered in the main paper.

2 Additional Results for the Simulation Experiments in Section 8

In this section, we provide additional information about the simulation experiments summarized in Section 8 of the main paper. Recall that we consider the $M_t/GI/s_t + M$ multi-server queueing model, having a nonhomogeneous Poisson arrival process (the M_t), i.i.d. service times distributed according to a random variable S with a general distribution (the GI), a time-varying staffing level (number of servers, the s_t) and customer abandonment with i.i.d. exponentially random patience times (the $+M$). The arrival process, service times and patience times are assumed to be mutually independent. Consistent with many call centers, we let the mean patience time be 2. We simulated each model specification using matlab, performing 100 replications.

In §2.1, we first describe the three types of nonhomogeneous Poisson arrival process used in our experiments and their arrival rate approximations by constant, linear and quadratic functions. We then explain the different service time distributions and staffing method we use and show the performance of different models in §2.2. Comparison of the performance of different estimators follows in §2.3. The performance of one estimator, $\bar{W}_{L,\lambda,q,p}(t)$, for H_2 service is explored further in §2.4. §2.5 gives additional simulation results when we do 1000, instead of 100 replications for the most variable H_2 service time distribution. As in the main paper, we look further into three special cases to gain more insights into these different estimators: longer service times in §2.6, decreasing arrival rate in §2.7 and sinusoidal arrival rate in §2.8.

2.1 The Three Arrival Rate Processes

As the actual arrival processes, we initially consider three nonhomogeneous Poisson arrival process, having constant ($\lambda(t) = 45$), linear ($\lambda(t) = 36 + 3t$) and quadratic ($\lambda(t) = 53.333 + 2.222t - 0.185t^2$) arrival rate functions. We assume that the system starts empty at time $t = -12$, and generate arrivals according to these processes over the interval $[-12, 12]$. Figure 1 shows the three different arrival rate functions. We generated these arrival processes by thinning a homogeneous arrival process with rate λ^* for $\lambda^* \geq \lambda(t)$, $-12 \leq t \leq 12$. The homogeneous Poisson process generates potential arrivals. We then let a potential arrival at time t be an actual arrival in the nonhomogeneous arrival process with probability $\lambda(t)/\lambda^*$.

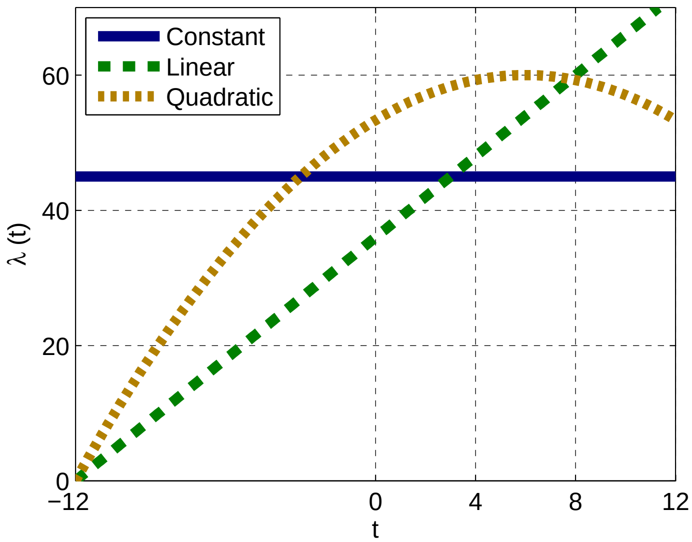

Figure 1: The three arrival rate functions used in simulation experiments: constant ($\lambda(t) = 45$), linear ($\lambda(t) = 36 + 3t$) and quadratic ($\lambda(t) = 53.333 + 2.222t - 0.185t^2$).

In any application, we fit candidate nonhomogeneous Poisson processes to the arrival data. Accordingly, with these three arrival processes, we also fit all three of the arrival processes, having constant, linear and quadratic arrival rate functions. Since the constant arrival rate function is a special case of the linear and quadratic arrival rate functions, for the constant arrival rate function, we see the impact of having more parameters than actually would be needed. On the other hand, for nonlinear quadratic arrival rate functions, we see the impact of fitting fewer parameters than needed. Presumably, fitting parameters to the specified process is best, but we see how important that is.

We generated 100 replications of each of the three time-varying arrival processes. Our goal is to compare the performance of different waiting time estimators over the interval $[0, t]$ for $t = 4$ and $t = 8$. Hence, in each replication, we need to approximate the arrival rate by constant, linear and quadratic functions over these subintervals. In order to better estimate the arrival rate function and to account for the time lag in the approximations, we estimate the arrival rate over $[-4, 4]$ and $[-8, 8]$ for the performance intervals $[0, 4]$ and $[0, 8]$, respectively.

Table 1 shows the average of the estimated parameters for the arrival rate functions over 100 replications in each case. When we approximate the arrival rate by constant function, we consider intervals $[0, 4]$ and $[0, 8]$ instead of $[-4, 4]$ and $[-8, 8]$. The halfwidths of 95% confidence intervals for all estimates are also reported.

		Constant	Linear		Quadratic		
Int.	*Arrival*	$\bar{\lambda}(t)$	a	b	a	b	c
$[-4, 4]$	*Constant*	45.2 ± 0.7	44.9 ± 0.5	0.099 ± 0.197	44.9 ± 0.7	0.099 ± 0.197	-0.013 ± 0.084
	Linear	41.7 ± 0.6	35.8 ± 0.5	2.907 ± 0.162	35.4 ± 0.6	2.907 ± 0.162	0.069 ± 0.083
	Quadratic	56.6 ± 0.7	52.1 ± 0.5	2.167 ± 0.212	52.8 ± 0.8	2.167 ± 0.212	-0.120 ± 0.112
$[-8, 8]$	*Constant*	45.1 ± 0.5	45.0 ± 0.4	0.017 ± 0.058	44.7 ± 0.6	0.017 ± 0.058	0.011 ± 0.018
	Linear	48.0 ± 0.5	35.9 ± 0.3	3.025 ± 0.064	35.6 ± 0.5	3.025 ± 0.064	0.016 ± 0.016
	Quadratic	58.3 ± 0.6	49.5 ± 0.4	2.185 ± 0.071	53.1 ± 0.5	2.185 ± 0.071	-0.167 ± 0.015

Table 1: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-4, 4]$ and $[-8, 8]$ to the arrival data for each arrival process; estimates with associated 95% confidence intervals based on 100 replications.

2.2 Average Performance of Different Models

We let the mean service time be $E[S] = 1$, so that we are measuring time in units of mean service times. We consider three service time distributions: exponential (M, having parameters $\gamma_W^2 = \theta_W^3 = 1$), Erlang E_4 (less variable, a sum of four i.i.d. exponentials, having parameters $\gamma_W^2 = 0.6125$ and $\theta_W^3 = 0.3125$) and hyperexponential H_2 (more variable, a mixture of two exponentials, having parameters $\gamma_W^2 = 3.0$ and $\theta_W^3 = 15.0$). The third H_2 parameter is chosen to produce balanced means as in (3.7) of [6]; the cdf is $P(S \le x) \equiv 1 - p_1 e^{-\lambda_1 x} - p_2 e^{-\lambda_2 x}$, where $p_1 = 0.0918$ and $p_2 = 0.9082$, while $\lambda_i = 2p_i$, yielding $p_i/\lambda_i = 1/2$ for $i = 1, 2$, $c^2 = 5$ and $E[S^3] = 90$.

The time-varying staffing is chosen to stabilize the performance at typical performance levels, following the method of [3] and [2]. In particular, the staffing is set using the square root staffing formula in (8.1), i.e., $s(t) \equiv \lceil m(t) + \beta\sqrt{m(t)} \rceil$, where $m(t)$ is the offered load and $\lceil x \rceil$ is the least integer greater than or equal to x. The offered load is $m(t) \equiv E[L(t)]$ in the associated IS model, which has formula (8) with the service time S playing the role of the waiting time W there. We consider three cases for the quality-of-service (QoS) parameter β: 0, 1 and 2. With abandonment in the model, the first two cases produce typical performance, while $\beta = 2$ corresponding to high QoS, producing performance close to the IS model. For the three arrival rate functions, we have explicit expressions for the offered load $m(t)$ and thus the staffing via (8.1). For the linear arrival rate function $36 + 3t$, the offered load is $m(t) = 36 + 3t - 3\gamma_W^2$. For the quadratic arrival rate, the offered load is $m(t) = 53.333 - 2.222\gamma_W^2 - 0.370\theta_W^3 + (2.222 + 0.370\gamma_W^2)t - 0.185t^2$. For the constant arrival rate, $m(t) = \lambda(t) = 45$.

As mentioned above, we generate 100 replications of each of the three time-varying arrival processes. For the service times, we again generate 100 replications of the three different distributions. We then combine each of the arrival process with service time process based on three different staffing with varying quality-of-service (QoS) parameter β in (8.1): 0, 1 and 2. This design gives us 27 different models in total, which is all of the possible combinations of the 3 arrival rate functions, 3 service time distributions and 3 QoS parameters. To minimize randomness in our experiments, we generate one set of patience times and use that for all models.

Table 2 shows the average performance of each of the models in terms of the average waiting time ($E[W]$), percent of arrivals delayed, and percent of arrivals abandoning. In each replication, we average the performance measures over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. The reported numbers are their averages over the 100 replications and the halfwidths of 95% confidence intervals.

Since we have staffed in order to stabilize performance, these performance measures should be close to corresponding steady-state values. Specifically, in Section 8.2, we analyzed the performance of constant arrival rate and exponential service times analytically using the algorithm described in [7]. Using the constant arrival rate $\lambda = 45$ and $E[S] = 1$, the stationary offered load is $m = \lambda E[S] = 45$, so that the staffing level with Qos parameter $\beta = 0$, 1 and 2 is $s = 45$, 52 and 59. In these three cases, the mean waiting time (in system) is 1.043, 1.0077 and 1.0008; the variance of the waiting time is 0.923, 0.938 and 0.986; the probability of delay is 0.602, 0.185 and 0.028; the probability of abandonment is 0.049, 0.0084 and 0.0008. Comparing these theoretical values to the experiment results in Table 2, we see that the performance is very similar; the numbers match almost exactly when we consider constant arrival rate and exponential service times (the first three rows of Table 2), but the performance in other models are also similar for the same value of the QoS parameter β.

We now further investigate whether the performance is indeed stabilized over the target intervals $[0, 4]$ and $[0, 8]$. Paralleling Figures 2-5 in the main paper, Figures 2 - 49 provide more information on the performance of different models over time. Performance of models with constant arrival rate is shown in Figures 2 - 13, with linear arrival rates in Figures 20 - 31 and with quadratic arrival rates in Figures 38 - 49. From these plots, we see that the performance is indeed typically stabilized

Performance Interval			[0, 4]			[0, 8]		
Arrival	*GI*	β	$E[W]$	*%Delayed*	*%Aban.*	$E[W]$	*%Delayed*	*%Aban.*
Constant	M	0	1.06 ± 0.02	61.3 ± 5.7	4.73 ± 0.94	1.08 ± 0.02	62.8 ± 4.8	4.18 ± 0.63
		1	1.01 ± 0.01	17.3 ± 4.2	0.89 ± 0.33	1.02 ± 0.01	16.8 ± 3.0	0.60 ± 0.19
		2	1.00 ± 0.01	3.6 ± 1.7	0.09 ± 0.08	1.01 ± 0.01	2.7 ± 1.0	0.05 ± 0.04
	E_4	0	1.05 ± 0.01	64.1 ± 4.9	4.09 ± 0.74	1.06 ± 0.01	63.0 ± 4.2	3.41 ± 0.48
		1	1.01 ± 0.01	17.1 ± 3.8	0.54 ± 0.19	1.01 ± 0.01	16.2 ± 2.6	0.38 ± 0.11
		2	1.00 ± 0.01	3.4 ± 1.4	0.03 ± 0.03	1.00 ± 0.01	2.5 ± 0.8	0.02 ± 0.02
	H_2	0	1.08 ± 0.04	48.6 ± 7.0	4.35 ± 1.11	1.08 ± 0.03	52.6 ± 5.9	4.20 ± 0.86
		1	1.03 ± 0.04	13.0 ± 4.2	0.68 ± 0.38	1.03 ± 0.03	14.5 ± 3.3	0.58 ± 0.24
		2	1.03 ± 0.03	2.2 ± 1.3	0.05 ± 0.04	1.02 ± 0.03	2.0 ± 0.9	0.03 ± 0.03
Linear	M	0	1.04 ± 0.02	52.3 ± 5.6	4.48 ± 0.88	1.05 ± 0.01	54.1 ± 4.4	3.80 ± 0.61
		1	1.00 ± 0.02	15.6 ± 3.7	0.70 ± 0.28	1.01 ± 0.01	16.4 ± 3.0	0.54 ± 0.16
		2	1.00 ± 0.02	2.4 ± 1.1	0.08 ± 0.06	1.00 ± 0.01	2.3 ± 0.9	0.05 ± 0.03
	E_4	0	1.04 ± 0.01	53.4 ± 5.1	3.91 ± 0.68	1.05 ± 0.01	56.1 ± 4.1	3.40 ± 0.47
		1	1.01 ± 0.01	14.7 ± 3.1	0.49 ± 0.18	1.01 ± 0.01	16.1 ± 2.4	0.42 ± 0.12
		2	1.01 ± 0.01	1.8 ± 0.8	0.05 ± 0.05	1.01 ± 0.00	2.3 ± 0.8	0.03 ± 0.03
	H_2	0	1.04 ± 0.04	52.3 ± 5.6	5.28 ± 1.11	1.06 ± 0.03	52.8 ± 4.7	4.56 ± 0.78
		1	1.01 ± 0.04	17.0 ± 4.2	1.00 ± 0.46	1.02 ± 0.03	18.4 ± 3.5	0.79 ± 0.29
		2	1.00 ± 0.04	3.5 ± 1.8	0.16 ± 0.11	1.01 ± 0.02	3.1 ± 1.3	0.09 ± 0.06
Quadratic	M	0	1.06 ± 0.02	56.4 ± 5.7	2.81 ± 0.54	1.06 ± 0.01	59.6 ± 4.6	3.31 ± 0.51
		1	1.02 ± 0.01	15.8 ± 3.6	0.33 ± 0.13	1.02 ± 0.01	17.7 ± 3.1	0.44 ± 0.12
		2	1.01 ± 0.01	1.8 ± 0.9	0.01 ± 0.01	1.01 ± 0.01	2.6 ± 0.9	0.03 ± 0.02
	E_4	0	1.05 ± 0.01	57.3 ± 5.4	2.43 ± 0.45	1.05 ± 0.01	59.8 ± 4.3	2.71 ± 0.41
		1	1.01 ± 0.01	14.1 ± 3.1	0.27 ± 0.12	1.01 ± 0.01	15.4 ± 2.7	0.32 ± 0.11
		2	1.00 ± 0.01	2.5 ± 1.4	0.02 ± 0.02	1.00 ± 0.00	2.6 ± 1.0	0.03 ± 0.02
	H_2	0	1.09 ± 0.04	61.7 ± 5.8	4.65 ± 0.94	1.08 ± 0.03	61.6 ± 5.3	4.98 ± 0.90
		1	1.03 ± 0.03	20.6 ± 4.1	0.71 ± 0.31	1.03 ± 0.02	22.3 ± 4.0	0.95 ± 0.34
		2	1.02 ± 0.03	3.6 ± 1.8	0.06 ± 0.05	1.02 ± 0.02	4.7 ± 1.8	0.13 ± 0.08

Table 2: Average performance of the 27 different $M_t/GI/s_t$ models, with given arrival rate function and staffing according to (8.1) with QoS parameter β averaged over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. Associated 95% confidence intervals based on 100 replications are also shown.

approximately by time $t = 0$. However, with H_2 service time distribution (where the service time is highly variable) the plots (for instance, see Figure 17) suggest that we might need more time to reach the steady state.

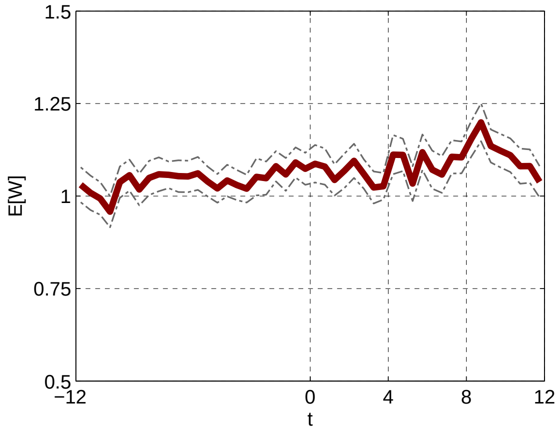

Figure 2: Average waiting time: QoS parameter $\beta = 0$.

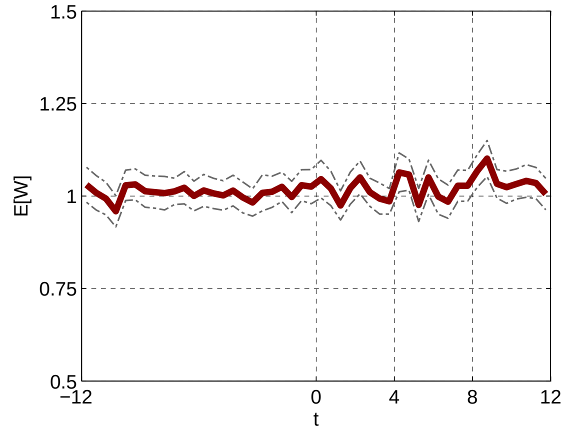

Figure 3: Average waiting time: QoS parameter $\beta = 1$.

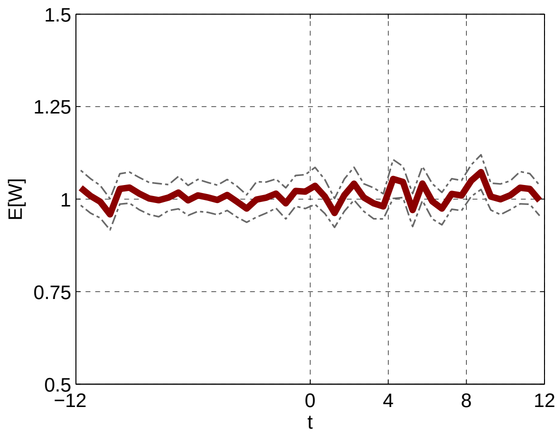

Figure 4: Average waiting time: QoS parameter $\beta = 2$.

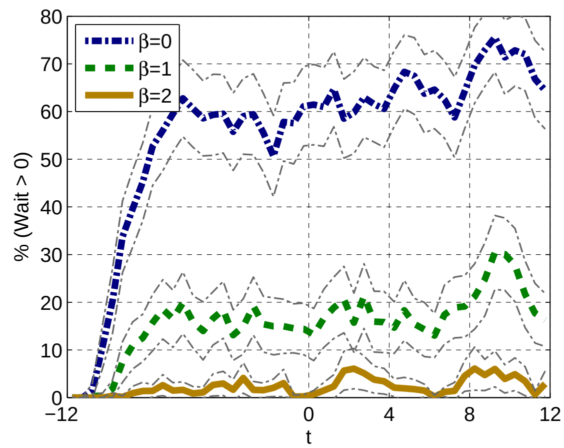

Figure 5: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

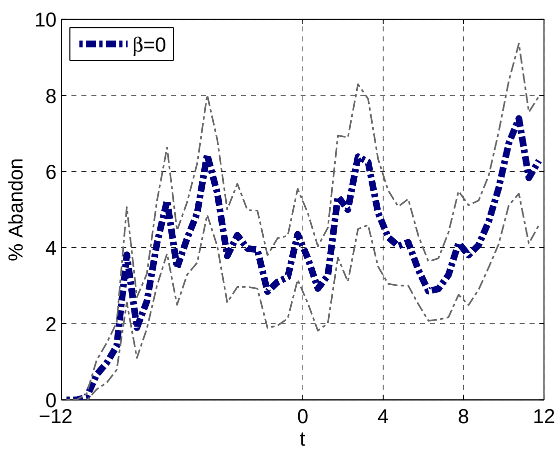

Figure 6: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

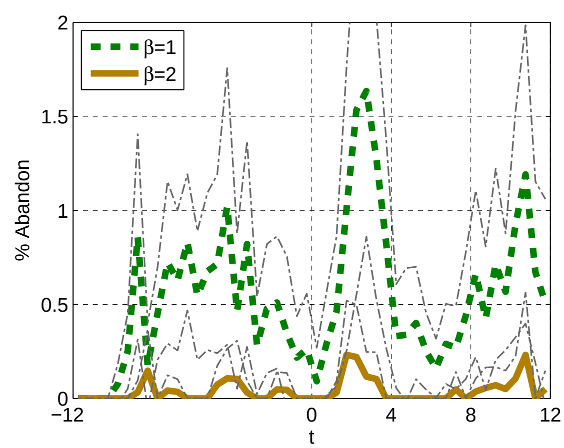

Figure 7: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 2-7: Constant arrival rate and M service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

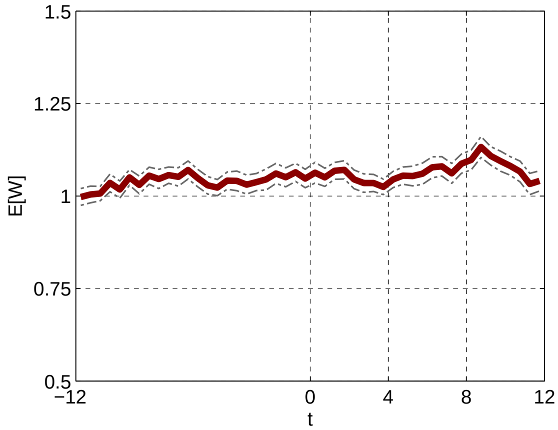

Figure 8: Average waiting time: QoS parameter $\beta = 0$.

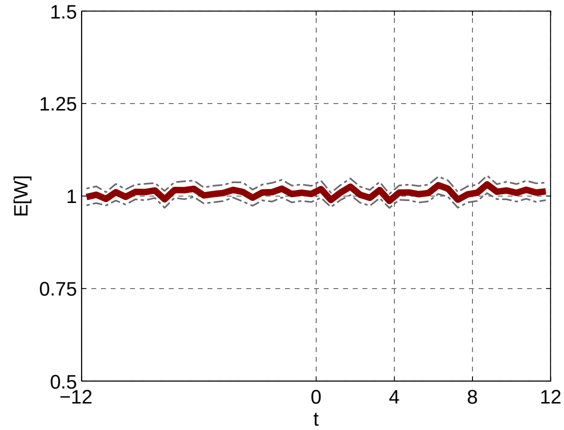

Figure 9: Average waiting time: QoS parameter $\beta = 1$.

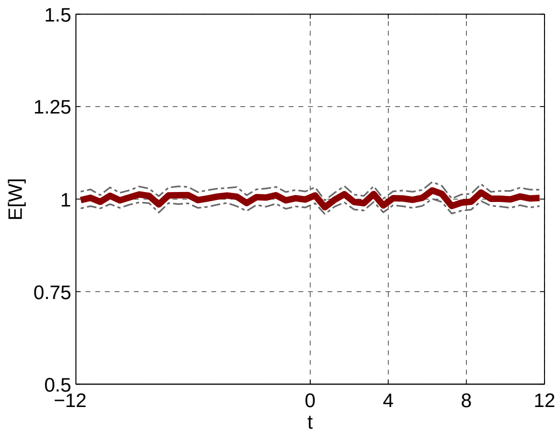

Figure 10: Average waiting time: QoS parameter $\beta = 2$.

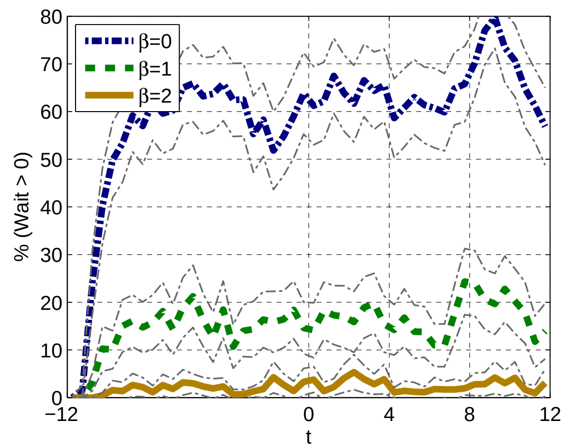

Figure 11: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

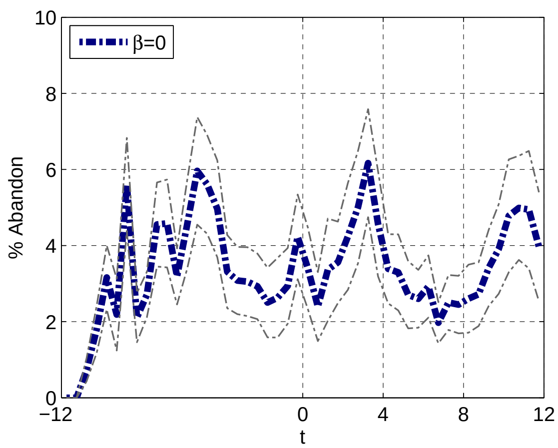

Figure 12: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

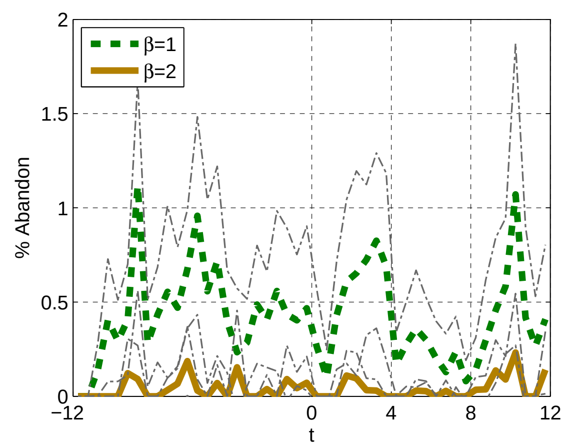

Figure 13: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 8-13: Constant arrival rate and E_4 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

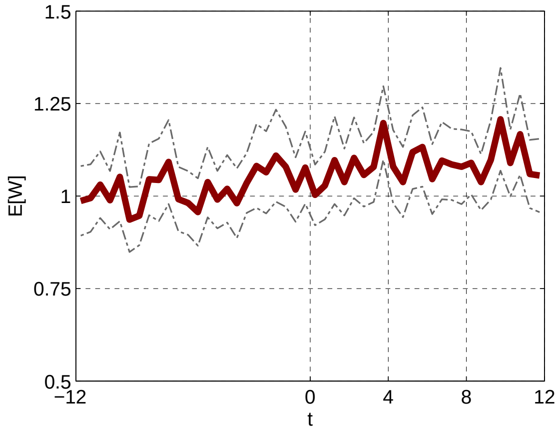

Figure 14: Average waiting time: QoS parameter $\beta = 0$.

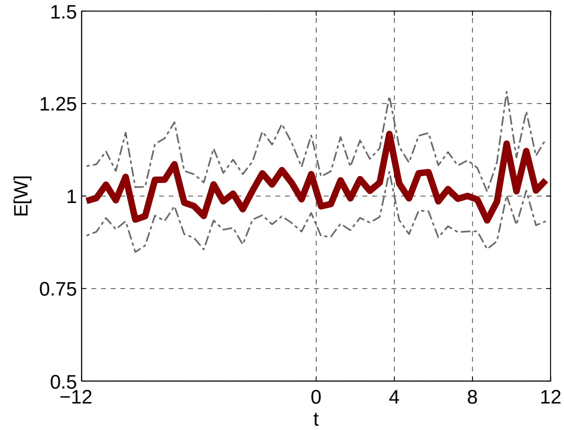

Figure 15: Average waiting time: QoS parameter $\beta = 1$.

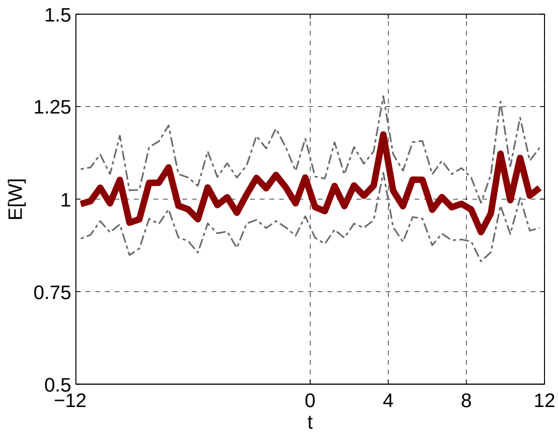

Figure 16: Average waiting time: QoS parameter $\beta = 2$.

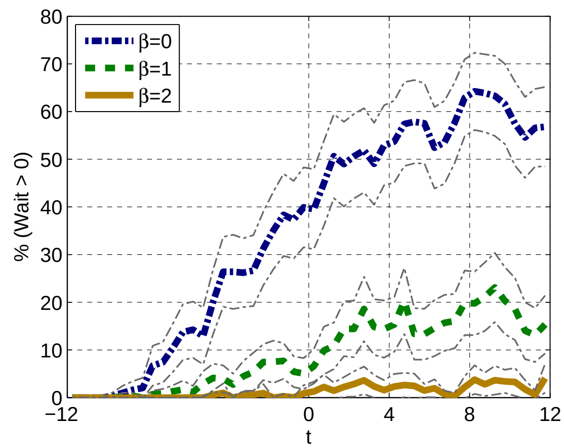

Figure 17: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

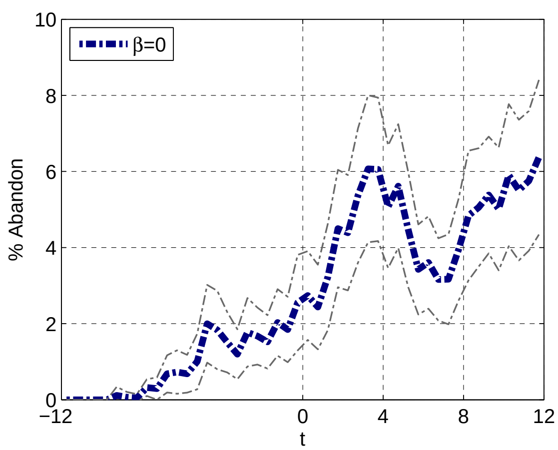

Figure 18: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

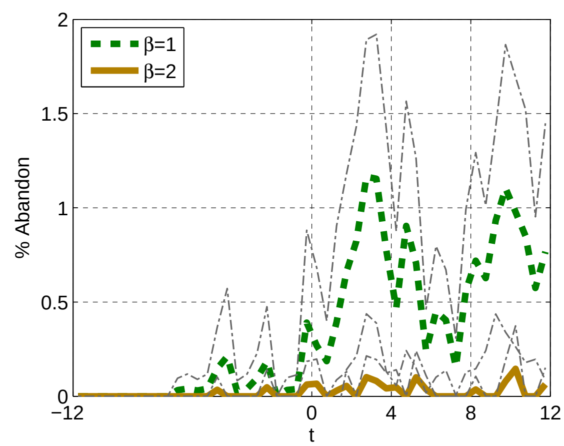

Figure 19: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 14-19: Constant arrival rate and H_2 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

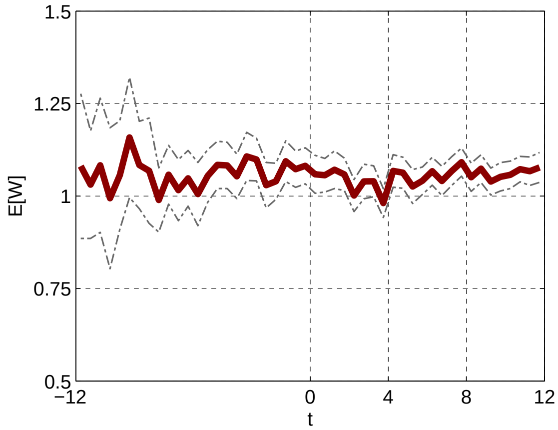

Figure 20: Average waiting time: QoS parameter $\beta = 0$.

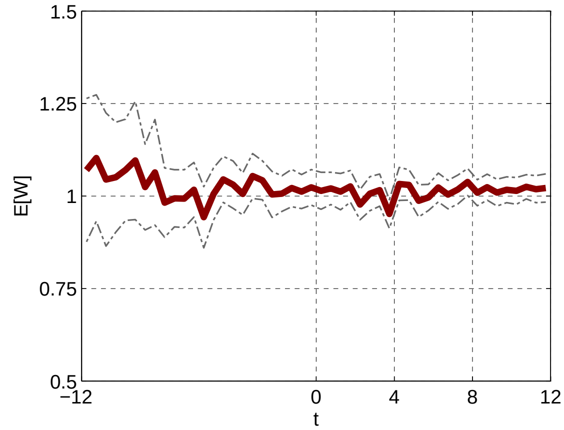

Figure 21: Average waiting time: QoS parameter $\beta = 1$.

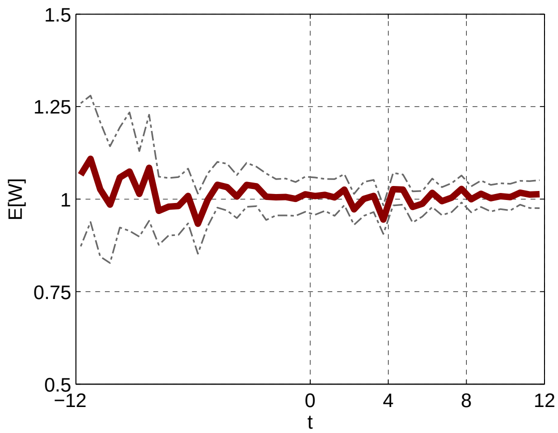

Figure 22: Average waiting time: QoS parameter $\beta = 2$.

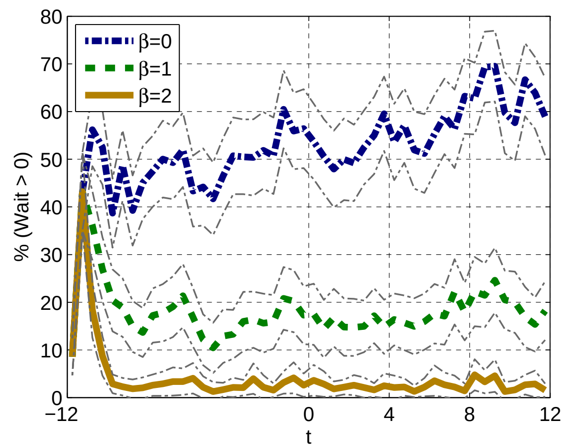

Figure 23: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

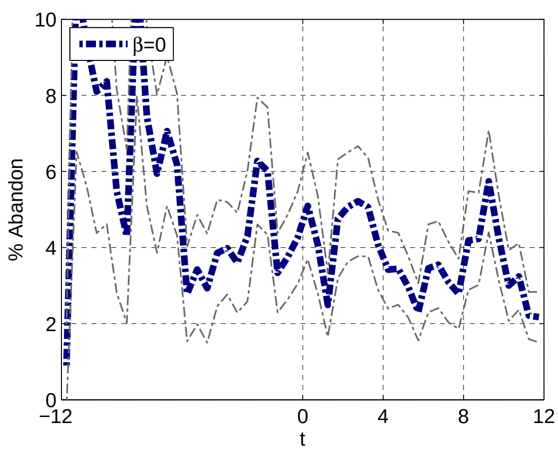

Figure 24: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

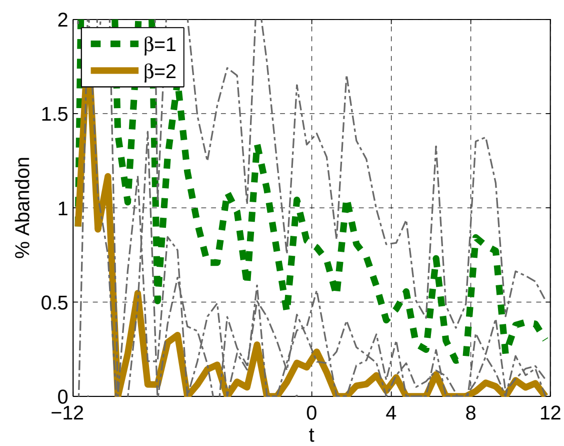

Figure 25: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 20-25: Linear arrival rate and M service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

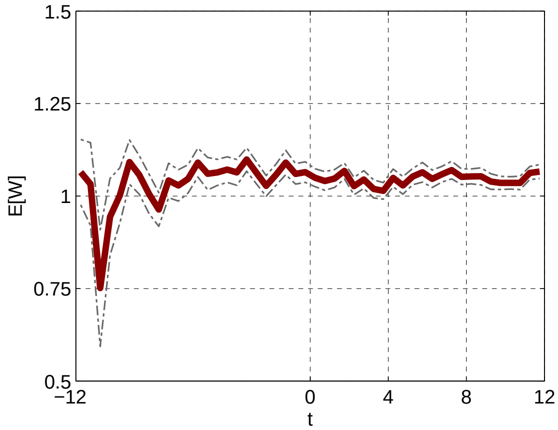

Figure 26: Average waiting time: QoS parameter $\beta = 0$.

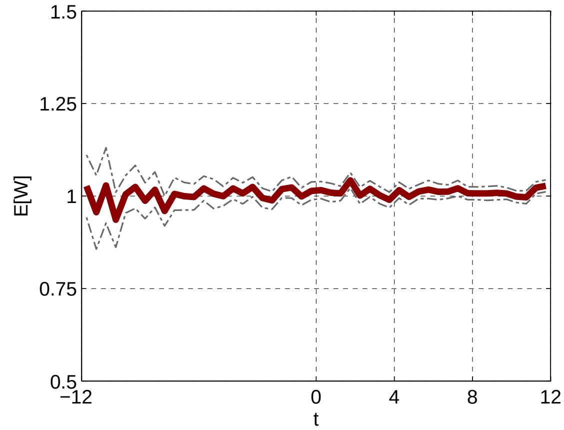

Figure 27: Average waiting time: QoS parameter $\beta = 1$.

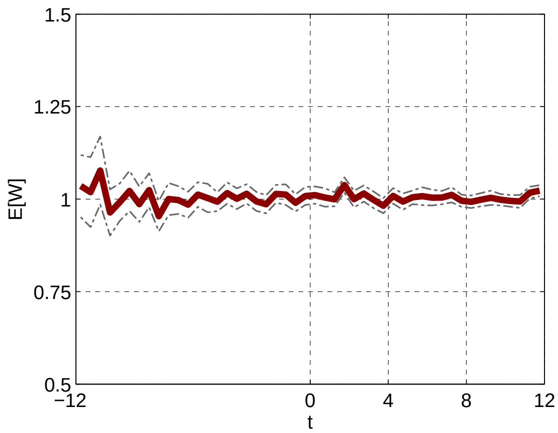

Figure 28: Average waiting time: QoS parameter $\beta = 2$.

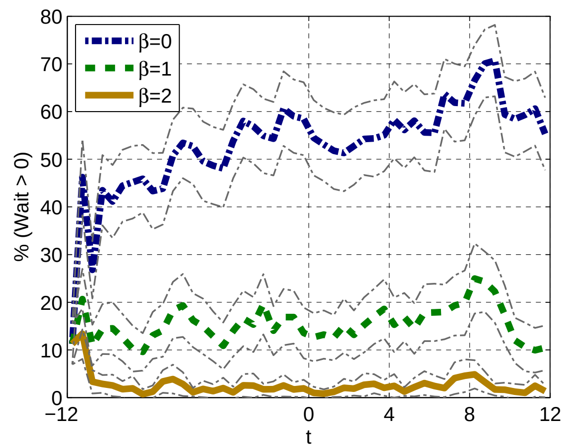

Figure 29: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

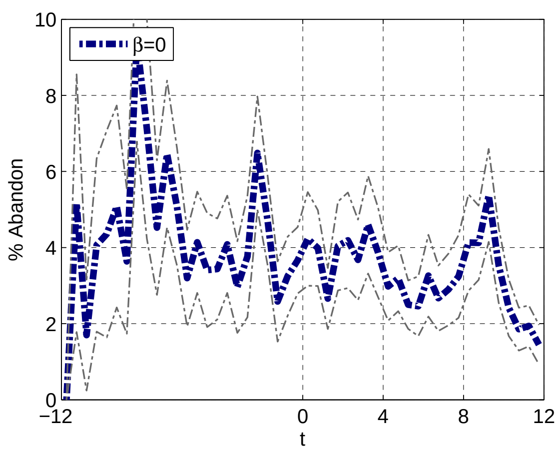

Figure 30: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

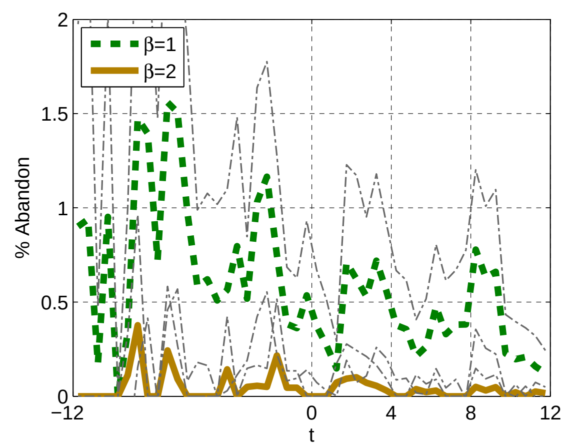

Figure 31: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 26-31: Linear arrival rate and E_4 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

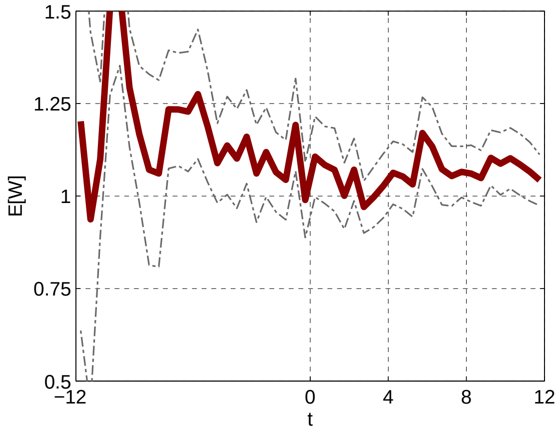

Figure 32: Average waiting time: QoS parameter $\beta = 0$.

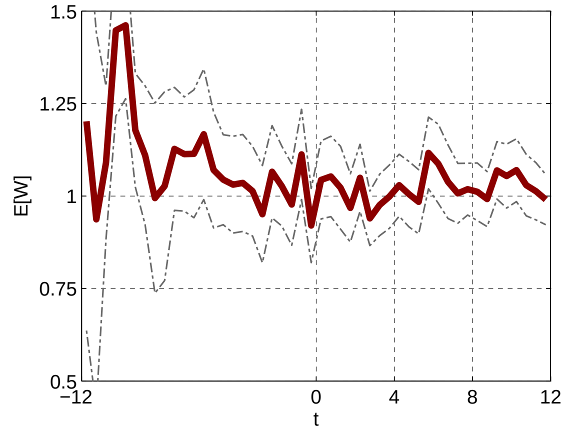

Figure 33: Average waiting time: QoS parameter $\beta = 1$.

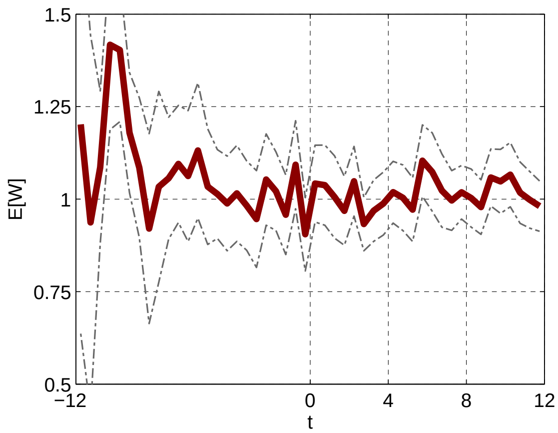

Figure 34: Average waiting time: QoS parameter $\beta = 2$.

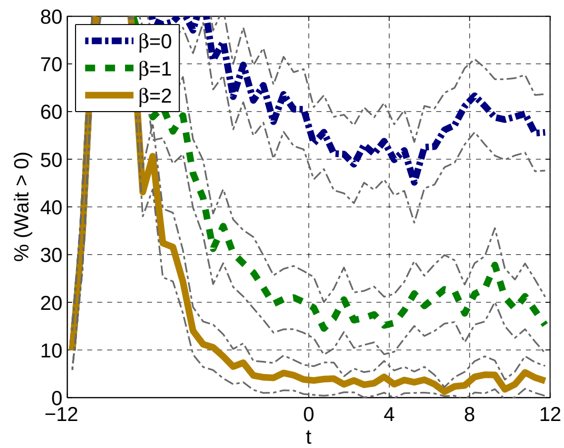

Figure 35: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

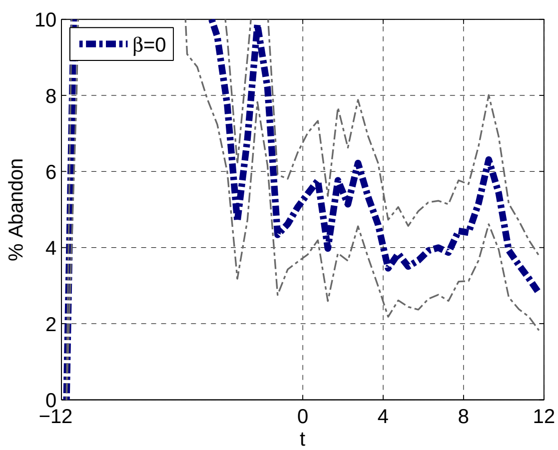

Figure 36: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

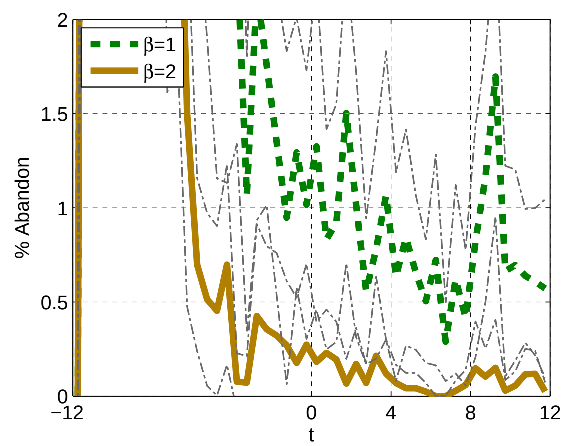

Figure 37: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 32-37: Linear arrival rate and H_2 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

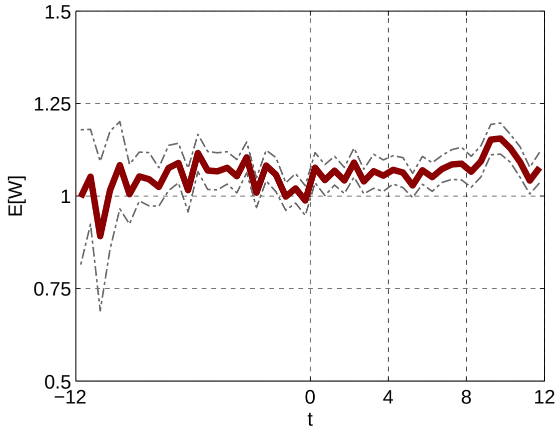

Figure 38: Average waiting time: QoS parameter $\beta = 0$.

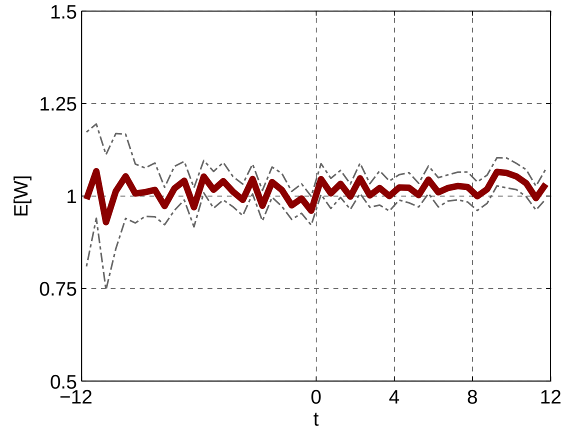

Figure 39: Average waiting time: QoS parameter $\beta = 1$.

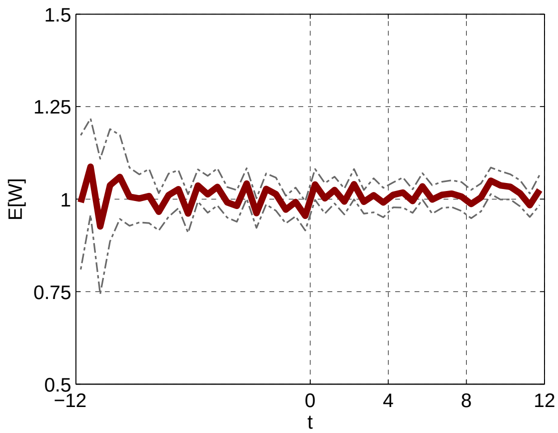

Figure 40: Average waiting time: QoS parameter $\beta = 2$.

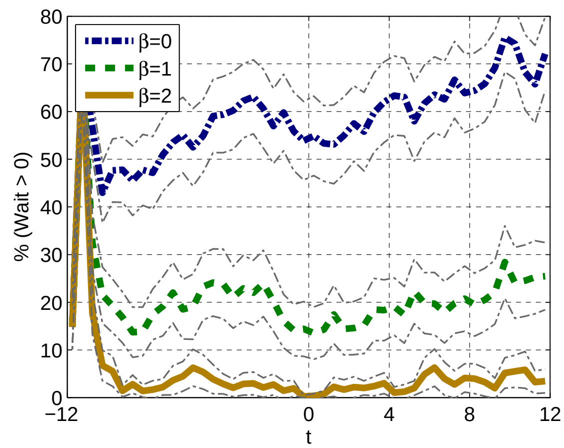

Figure 41: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

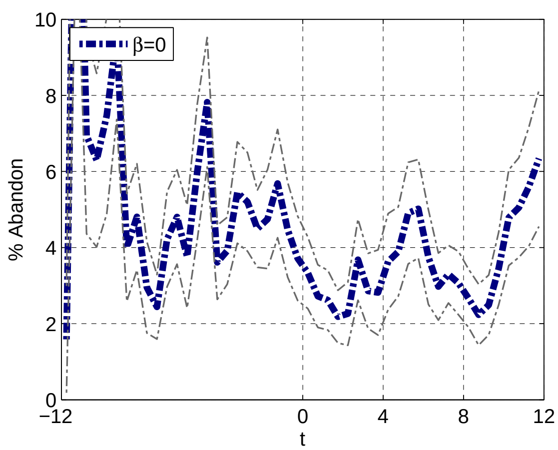

Figure 42: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

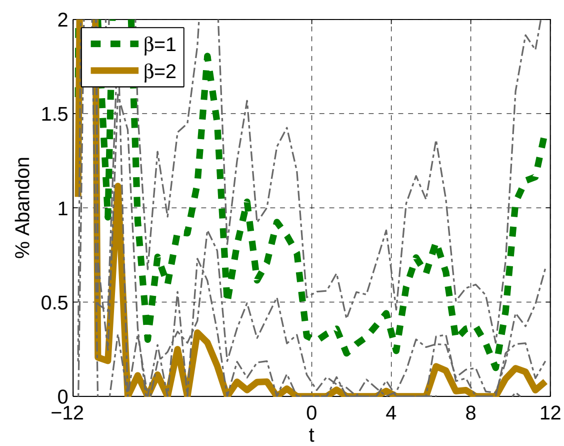

Figure 43: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 38-43: Quadratic arrival rate and M service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

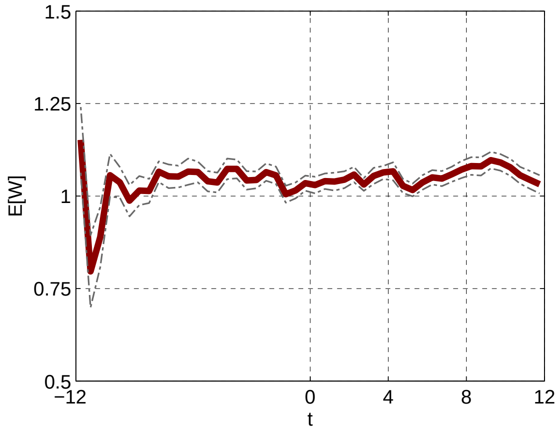

Figure 44: Average waiting time: QoS parameter $\beta = 0$.

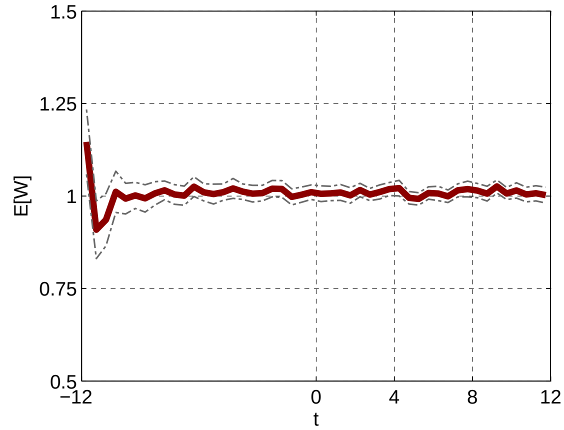

Figure 45: Average waiting time: QoS parameter $\beta = 1$.

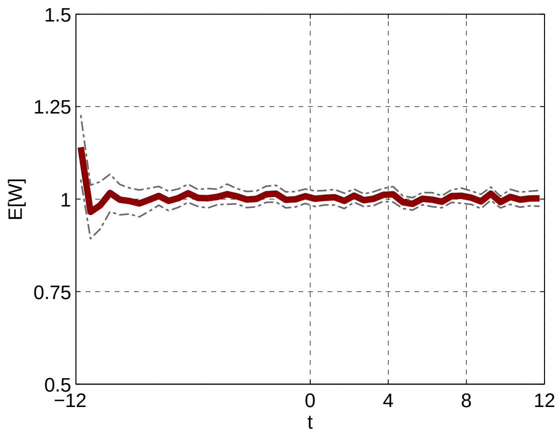

Figure 46: Average waiting time: QoS parameter $\beta = 2$.

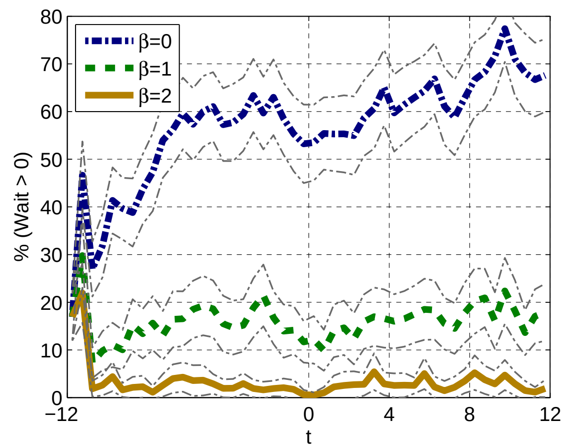

Figure 47: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

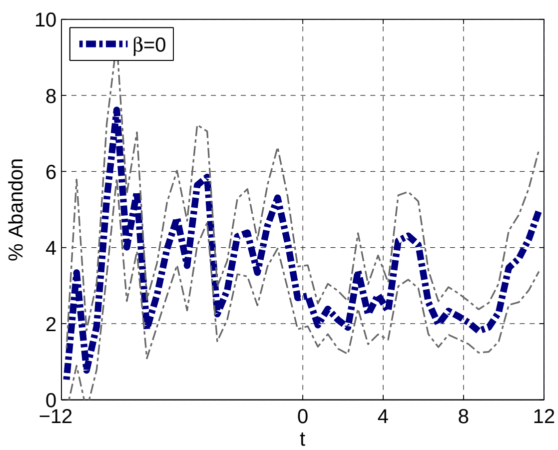

Figure 48: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

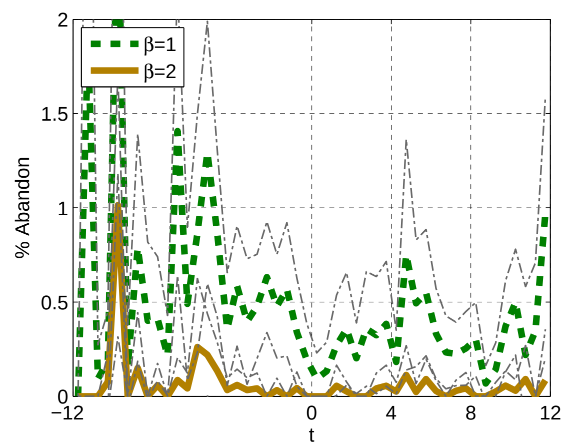

Figure 49: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 44-49: Quadratic arrival rate and E_4 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

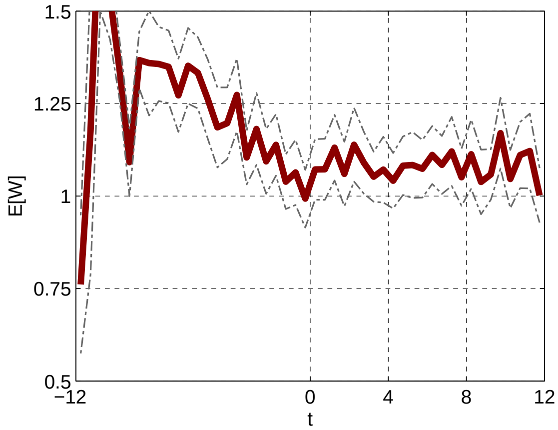

Figure 50: Average waiting time: QoS parameter $\beta = 0$.

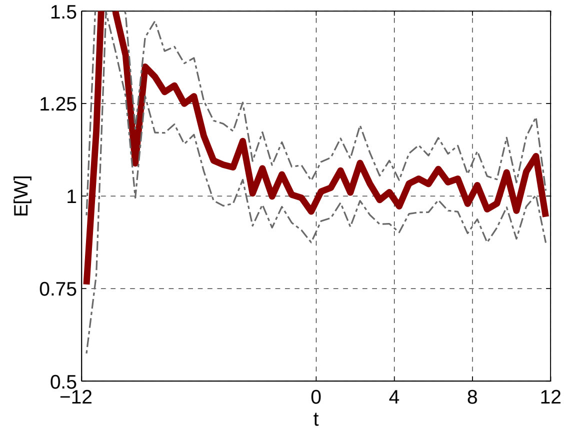

Figure 51: Average waiting time: QoS parameter $\beta = 1$.

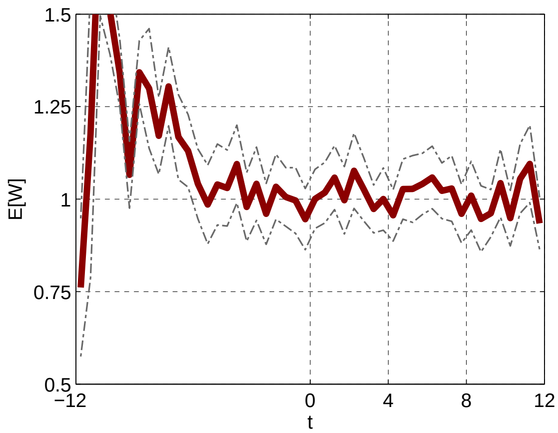

Figure 52: Average waiting time: QoS parameter $\beta = 2$.

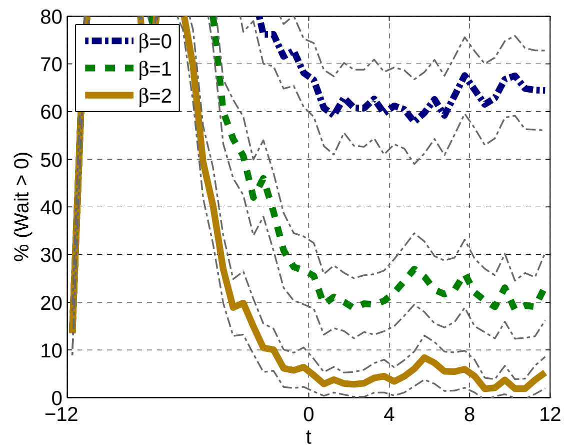

Figure 53: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

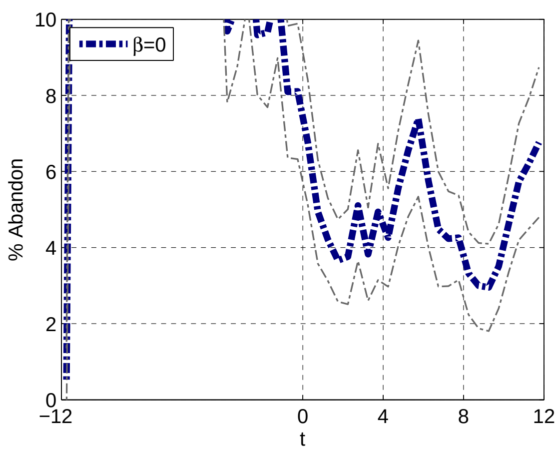

Figure 54: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

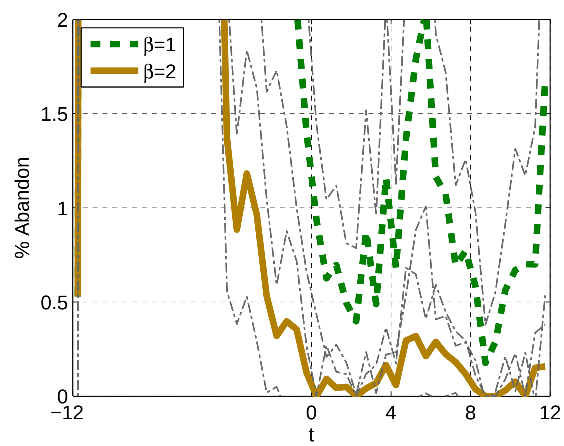

Figure 55: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 50-55: Quadratic arrival rate and H_2 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

2.3 Estimation Results

We now describe the estimation results. We consider ten different methods for the constant, linear and quadratic arrival rate functions. The first estimator is the direct average $\bar{W}(t)$ in (1.1), which we could not use if the waiting times were not actually observed. The second is the indirect estimator $\bar{W}_{L,\lambda}(t)$ in (1.2) based on LL, whose bias we want to reduce. Then we give the estimators $\bar{W}_{L,\lambda,r}(t)$ in (1.4) from [4] and its extension $\bar{W}_{L,\lambda,r,\gamma}(t)$ in (6.3) of the main paper, which are based on the sample path relation in (1.3) of [4]. Next we consider estimators based on the TVLL. We do not consider the direct estimator $\bar{W}_{tvll}(t)$ provided by Theorem 2 here, because we consider it covered by the linear approximation, as we demonstrated in §8.5. For the TVLL-based estimators, we consider the estimator $\bar{W}_{L,\lambda,l}(t)$ from §4 based on the fitted linear arrival rate function, its perturbation refinement $\bar{W}_{L,\lambda,l,p}(t)$ from §5 and the estimated best of these two, $\bar{W}_{L,\lambda,l,b}(t)$, chosen as the one with the smaller confidence interval. Finally there are the corresponding three estimators from §7 based on the fitted quadratic arrival rate function.

Since we consider multi-server queues with reasonable staffing (specified below), the waiting times (time spent in system) do not differ greatly from the service times. For customers that are served, the waiting times are somewhat longer because of the time spent in queue, but that usually is relatively short compared to the service times. Longer waiting times in queue are reduced by customer abandonment. Thus, in our TVLL linear and quadratic estimation procedures, we approximate the unknown (γ_W^2, θ_W^3) by the specified (γ_S^2, θ_S^3).

Tables 3 - 5 show the estimated waiting times by these ten methods.

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.062 ± 0.019	1.066 ± 0.019	1.064 ± 0.019	1.064 ± 0.019	1.069 ± 0.019	1.065 ± 0.018	1.065 ± 0.018	1.068 ± 0.018	1.062 ± 0.017	1.062 ± 0.017
		1	1.014 ± 0.015	1.015 ± 0.015	1.017 ± 0.015	1.017 ± 0.015	1.018 ± 0.015	1.015 ± 0.015	1.015 ± 0.015	1.017 ± 0.014	1.013 ± 0.014	1.013 ± 0.014
		2	1.005 ± 0.013	1.005 ± 0.014	1.006 ± 0.013	1.006 ± 0.013	1.008 ± 0.013	1.005 ± 0.013	1.008 ± 0.013	1.007 ± 0.013	1.003 ± 0.013	1.007 ± 0.013
	[0, 8]	0	1.077 ± 0.017	1.068 ± 0.016	1.079 ± 0.018	1.079 ± 0.018	1.070 ± 0.016	1.068 ± 0.016	1.068 ± 0.016	1.072 ± 0.016	1.069 ± 0.015	1.069 ± 0.015
		1	1.019 ± 0.012	1.014 ± 0.012	1.020 ± 0.012	1.020 ± 0.012	1.016 ± 0.011	1.014 ± 0.011	1.016 ± 0.011	1.017 ± 0.011	1.015 ± 0.011	1.017 ± 0.011
		2	1.009 ± 0.011	1.005 ± 0.011	1.010 ± 0.011	1.010 ± 0.011	1.006 ± 0.010	1.005 ± 0.011	1.006 ± 0.010	1.008 ± 0.010	1.006 ± 0.010	1.008 ± 0.010
	Avg		1.031	1.029	1.033	1.033	1.031	1.029	1.030	1.032	1.028	1.030
E_4	[0, 4]	0	1.052 ± 0.011	1.058 ± 0.014	1.051 ± 0.013	1.054 ± 0.012	1.060 ± 0.014	1.058 ± 0.014	1.058 ± 0.014	1.059 ± 0.012	1.056 ± 0.011	1.056 ± 0.011
		1	1.008 ± 0.008	1.011 ± 0.010	1.007 ± 0.010	1.008 ± 0.009	1.014 ± 0.010	1.012 ± 0.010	1.012 ± 0.010	1.013 ± 0.009	1.010 ± 0.008	1.010 ± 0.008
		2	0.999 ± 0.007	1.002 ± 0.009	0.998 ± 0.010	0.999 ± 0.008	1.004 ± 0.009	1.002 ± 0.008	1.002 ± 0.008	1.004 ± 0.007	1.001 ± 0.007	1.001 ± 0.007
	[0, 8]	0	1.060 ± 0.010	1.054 ± 0.010	1.061 ± 0.011	1.058 ± 0.010	1.056 ± 0.011	1.054 ± 0.010	1.054 ± 0.010	1.057 ± 0.011	1.055 ± 0.010	1.055 ± 0.010
		1	1.009 ± 0.006	1.007 ± 0.007	1.010 ± 0.007	1.009 ± 0.006	1.009 ± 0.008	1.007 ± 0.007	1.007 ± 0.007	1.010 ± 0.008	1.008 ± 0.006	1.008 ± 0.006
		2	1.001 ± 0.005	0.999 ± 0.006	1.002 ± 0.006	1.001 ± 0.006	1.001 ± 0.007	0.999 ± 0.006	0.999 ± 0.006	1.002 ± 0.007	1.000 ± 0.005	1.000 ± 0.005
	Avg		1.021	1.022	1.021	1.022	1.024	1.022	1.022	1.024	1.021	1.021
H_2	[0, 4]	0	1.069 ± 0.036	1.020 ± 0.031	1.036 ± 0.033	1.068 ± 0.046	1.033 ± 0.034	1.021 ± 0.033	1.021 ± 0.033	0.201 ± 0.273	0.965 ± 0.063	0.965 ± 0.063
		1	1.023 ± 0.032	0.974 ± 0.024	0.986 ± 0.024	1.010 ± 0.034	0.984 ± 0.025	0.974 ± 0.024	0.974 ± 0.024	0.281 ± 0.255	0.927 ± 0.049	0.927 ± 0.049
		2	1.019 ± 0.032	0.966 ± 0.021	0.977 ± 0.022	0.999 ± 0.032	0.977 ± 0.022	0.967 ± 0.022	0.967 ± 0.022	0.277 ± 0.254	0.919 ± 0.046	0.919 ± 0.046
	[0, 8]	0	1.078 ± 0.031	1.040 ± 0.026	1.058 ± 0.027	1.093 ± 0.035	1.043 ± 0.027	1.040 ± 0.026	1.040 ± 0.026	1.047 ± 0.029	1.031 ± 0.026	1.031 ± 0.026
		1	1.022 ± 0.026	0.988 ± 0.019	0.998 ± 0.019	1.018 ± 0.025	0.990 ± 0.019	0.987 ± 0.019	0.987 ± 0.019	0.992 ± 0.021	0.980 ± 0.019	0.980 ± 0.019
		2	1.013 ± 0.025	0.979 ± 0.017	0.987 ± 0.017	1.004 ± 0.021	0.981 ± 0.017	0.979 ± 0.017	0.979 ± 0.017	0.982 ± 0.019	0.972 ± 0.017	0.972 ± 0.017
	Avg		1.037	0.994	1.007	1.032	1.001	0.995	0.995	0.630	0.966	0.966

Table 3: CONSTANT arrival rate: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.038 ± 0.019	0.980 ± 0.020	1.047 ± 0.020	1.047 ± 0.020	1.058 ± 0.023	1.046 ± 0.022	1.046 ± 0.022	1.062 ± 0.021	1.046 ± 0.021	1.046 ± 0.021
		1	1.002 ± 0.016	0.939 ± 0.015	1.005 ± 0.015	1.005 ± 0.015	1.011 ± 0.016	1.000 ± 0.016	1.011 ± 0.016	1.014 ± 0.015	1.000 ± 0.015	1.014 ± 0.015
		2	0.996 ± 0.016	0.933 ± 0.014	1.000 ± 0.014	1.000 ± 0.014	1.003 ± 0.015	0.993 ± 0.015	1.003 ± 0.015	1.007 ± 0.013	0.993 ± 0.014	1.007 ± 0.013
	[0, 8]	0	1.051 ± 0.013	0.983 ± 0.015	1.050 ± 0.015	1.050 ± 0.015	1.052 ± 0.017	1.044 ± 0.016	1.044 ± 0.016	1.054 ± 0.016	1.045 ± 0.016	1.045 ± 0.016
		1	1.010 ± 0.010	0.944 ± 0.011	1.006 ± 0.011	1.006 ± 0.011	1.008 ± 0.012	1.001 ± 0.012	1.001 ± 0.012	1.009 ± 0.011	1.002 ± 0.011	1.009 ± 0.011
		2	1.003 ± 0.009	0.938 ± 0.010	0.998 ± 0.010	0.998 ± 0.010	1.000 ± 0.011	0.993 ± 0.011	0.993 ± 0.011	1.002 ± 0.010	0.994 ± 0.010	1.002 ± 0.010
	Avg		1.017	0.953	1.018	1.018	1.022	1.013	1.016	1.025	1.013	1.021
E_4	[0, 4]	0	1.039 ± 0.009	0.997 ± 0.012	1.069 ± 0.010	1.042 ± 0.011	1.045 ± 0.013	1.040 ± 0.013	1.040 ± 0.013	1.048 ± 0.012	1.041 ± 0.012	1.048 ± 0.012
		1	1.010 ± 0.008	0.963 ± 0.010	1.039 ± 0.008	1.011 ± 0.008	1.008 ± 0.010	1.004 ± 0.010	1.008 ± 0.010	1.012 ± 0.009	1.005 ± 0.009	1.012 ± 0.009
		2	1.005 ± 0.007	0.959 ± 0.009	1.033 ± 0.008	1.005 ± 0.008	1.003 ± 0.010	0.998 ± 0.010	1.003 ± 0.010	1.006 ± 0.008	1.000 ± 0.009	1.006 ± 0.008
	[0, 8]	0	1.048 ± 0.008	1.001 ± 0.008	1.070 ± 0.008	1.044 ± 0.008	1.043 ± 0.010	1.040 ± 0.009	1.040 ± 0.009	1.044 ± 0.009	1.041 ± 0.008	1.041 ± 0.008
		1	1.011 ± 0.005	0.967 ± 0.005	1.033 ± 0.006	1.008 ± 0.005	1.006 ± 0.006	1.003 ± 0.005	1.003 ± 0.005	1.007 ± 0.006	1.005 ± 0.005	1.005 ± 0.005
		2	1.004 ± 0.005	0.960 ± 0.005	1.026 ± 0.005	1.001 ± 0.005	0.999 ± 0.005	0.997 ± 0.005	0.997 ± 0.005	1.000 ± 0.005	0.998 ± 0.005	0.998 ± 0.005
	Avg		1.020	0.974	1.045	1.018	1.017	1.014	1.015	1.020	1.015	1.018
H_2	[0, 4]	0	1.041 ± 0.035	0.854 ± 0.026	0.909 ± 0.029	1.017 ± 0.043	1.157 ± 0.061	1.007 ± 0.036	1.007 ± 0.036	0.348 ± 1.511	1.230 ± 0.568	1.230 ± 0.568
		1	1.006 ± 0.035	0.811 ± 0.020	0.868 ± 0.021	0.981 ± 0.033	1.058 ± 0.041	0.948 ± 0.027	0.948 ± 0.027	0.567 ± 1.488	-11.3 ± 24.2	0.567 ± 1.488
		2	0.998 ± 0.035	0.802 ± 0.017	0.858 ± 0.018	0.971 ± 0.028	1.043 ± 0.038	0.935 ± 0.021	0.935 ± 0.021	0.520 ± 1.490	1.444 ± 1.305	1.444 ± 1.305
	[0, 8]	0	1.063 ± 0.027	0.873 ± 0.019	0.931 ± 0.022	1.048 ± 0.029	1.116 ± 0.040	1.018 ± 0.026	1.018 ± 0.026	0.852 ± 0.244	1.009 ± 0.026	1.009 ± 0.026
		1	1.021 ± 0.026	0.831 ± 0.014	0.884 ± 0.015	0.991 ± 0.021	1.038 ± 0.027	0.962 ± 0.019	0.962 ± 0.019	0.853 ± 0.202	0.954 ± 0.019	0.954 ± 0.019
		2	1.013 ± 0.025	0.822 ± 0.012	0.874 ± 0.013	0.980 ± 0.018	1.020 ± 0.021	0.949 ± 0.015	0.949 ± 0.015	0.970 ± 0.095	0.942 ± 0.015	0.942 ± 0.015
	Avg		1.024	0.832	0.887	0.998	1.072	0.970	0.970	0.685	-0.959	1.024

Table 4: LINEAR arrival rate: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

GI	*Int*	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.059 ± 0.015	1.014 ± 0.015	1.056 ± 0.016	1.056 ± 0.016	1.058 ± 0.017	1.052 ± 0.016	1.052 ± 0.016	1.053 ± 0.015	1.046 ± 0.014	1.046 ± 0.014
		1	1.017 ± 0.012	0.977 ± 0.011	1.014 ± 0.012	1.014 ± 0.012	1.018 ± 0.013	1.013 ± 0.012	1.013 ± 0.012	1.013 ± 0.011	1.007 ± 0.011	1.007 ± 0.011
		2	1.009 ± 0.012	0.971 ± 0.011	1.006 ± 0.012	1.006 ± 0.012	1.011 ± 0.012	1.006 ± 0.011	1.006 ± 0.011	1.006 ± 0.010	1.000 ± 0.010	1.000 ± 0.010
	[0, 8]	0	1.064 ± 0.013	1.033 ± 0.012	1.059 ± 0.013	1.059 ± 0.013	1.077 ± 0.014	1.073 ± 0.013	1.073 ± 0.013	1.056 ± 0.013	1.054 ± 0.013	1.054 ± 0.013
		1	1.020 ± 0.010	0.993 ± 0.009	1.015 ± 0.010	1.015 ± 0.010	1.034 ± 0.010	1.030 ± 0.010	1.030 ± 0.010	1.015 ± 0.010	1.013 ± 0.009	1.013 ± 0.009
		2	1.010 ± 0.009	0.985 ± 0.008	1.006 ± 0.009	1.006 ± 0.009	1.025 ± 0.009	1.021 ± 0.008	1.021 ± 0.008	1.006 ± 0.009	1.005 ± 0.008	1.005 ± 0.008
	Avg		1.030	0.995	1.026	1.026	1.037	1.033	1.033	1.025	1.021	1.021
E_4	[0, 4]	0	1.045 ± 0.010	1.020 ± 0.011	1.057 ± 0.012	1.043 ± 0.011	1.047 ± 0.012	1.044 ± 0.011	1.044 ± 0.011	1.042 ± 0.010	1.039 ± 0.010	1.039 ± 0.010
		1	1.009 ± 0.007	0.989 ± 0.008	1.019 ± 0.009	1.008 ± 0.008	1.014 ± 0.009	1.012 ± 0.009	1.012 ± 0.009	1.010 ± 0.008	1.007 ± 0.008	1.007 ± 0.008
		2	1.002 ± 0.007	0.983 ± 0.008	1.012 ± 0.008	1.001 ± 0.007	1.009 ± 0.008	1.006 ± 0.008	1.006 ± 0.008	1.004 ± 0.007	1.001 ± 0.007	1.004 ± 0.007
	[0, 8]	0	1.048 ± 0.008	1.031 ± 0.008	1.055 ± 0.009	1.046 ± 0.008	1.058 ± 0.010	1.056 ± 0.009	1.056 ± 0.009	1.043 ± 0.009	1.042 ± 0.008	1.042 ± 0.008
		1	1.009 ± 0.005	0.996 ± 0.006	1.015 ± 0.006	1.008 ± 0.006	1.021 ± 0.007	1.019 ± 0.006	1.019 ± 0.006	1.007 ± 0.006	1.006 ± 0.006	1.006 ± 0.006
		2	1.002 ± 0.004	0.989 ± 0.005	1.008 ± 0.005	1.001 ± 0.005	1.014 ± 0.006	1.012 ± 0.005	1.012 ± 0.005	1.001 ± 0.006	1.000 ± 0.005	1.000 ± 0.005
	Avg		1.019	1.001	1.028	1.018	1.027	1.025	1.025	1.018	1.016	1.016
H_2	[0, 4]	0	1.087 ± 0.036	0.914 ± 0.025	0.948 ± 0.027	1.016 ± 0.042	1.048 ± 0.035	1.008 ± 0.029	1.008 ± 0.029	-0.265 ± 0.316	0.985 ± 0.072	0.985 ± 0.072
		1	1.031 ± 0.034	0.860 ± 0.017	0.895 ± 0.019	0.963 ± 0.032	0.973 ± 0.022	0.942 ± 0.019	0.942 ± 0.019	-0.144 ± 0.298	0.916 ± 0.053	0.916 ± 0.053
		2	1.020 ± 0.033	0.854 ± 0.015	0.889 ± 0.017	0.959 ± 0.026	0.964 ± 0.020	0.935 ± 0.017	0.935 ± 0.017	-0.147 ± 0.297	0.907 ± 0.051	0.907 ± 0.051
	[0, 8]	0	1.088 ± 0.027	0.954 ± 0.023	0.985 ± 0.024	1.048 ± 0.031	1.091 ± 0.031	1.057 ± 0.027	1.057 ± 0.027	0.537 ± 0.294	1.078 ± 0.031	1.078 ± 0.031
		1	1.032 ± 0.024	0.903 ± 0.017	0.931 ± 0.018	0.988 ± 0.023	1.021 ± 0.021	0.994 ± 0.020	0.994 ± 0.020	0.931 ± 0.140	1.007 ± 0.022	1.007 ± 0.022
		2	1.021 ± 0.023	0.894 ± 0.014	0.921 ± 0.015	0.976 ± 0.019	1.009 ± 0.018	0.983 ± 0.017	0.983 ± 0.017	0.960 ± 0.114	0.995 ± 0.019	0.995 ± 0.019
	Avg		1.046	0.897	0.928	0.992	1.018	0.987	0.987	0.312	0.981	0.981

Table 5: QUADRATIC arrival rate: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

Tables 6, 7 and 8 provide additional information: the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (4.6) and (7.6)). We can again compare our simulation results with the theoretical reference points discussed in Section 8.2. We note that the linear arrival rate case is like the IS reference linear case. We computed that for the M, E_4 and H_2 service time distributions, respectively, we have $E[\bar{L}(t)] = 39.0$, 40.125 and 33.0 over $[0,4]$ and 45, 46.125 and 39.0 over $[0,8]$. The indirect estimator $\bar{W}_{L,\lambda}(t)$ in (1.2) takes the values $39/42 = 0.929$, $40.125/42 = 0.955$ and $33/42 = 0.786$ over $[0,4]$, and $45/48 = 0.938$, $46.125/48 = 0.961$ and $39.0/48 = 0.813$ over $[0,8]$. These numbers translate to the estimation bias in $\bar{W}_{L,\lambda}(t)$ of 7.1%, 4.5% and 21.4% over $[0,4]$, and 6.2%, 3.9% and 18.7% over $[0,8]$. The estimated bias in Table 7 closely matches the results for the M, E_4 and H_2 service with $\beta = 2$, which is very similar to the infinite server (IS) case. Similarly, for the quadratic arrival rate function, we have $E[\bar{L}(t)] = 56.173$, 58.140 and 55.432 over $[0,4]$ and 62.840, 65.363 and 60.617 over $[0,8]$ for the M, E_4 and H_2 service time distributions, respectively. The indirect estimator $\bar{W}_{L,\lambda}(t)$ takes the values $56.173/58.765 = 0.956$, $58.140/58.765 = 0.989$ and $55.432/58.765 = 0.943$ over $[0,4]$, $62.840/66.173 = 0.950$, $65.363/66.173 = 0.988$ and $60.617/66.173 = 0.916$ over $[0,8]$. That means that the estimation bias in $\bar{W}_{L,\lambda}(t)$ is 4.4%, 1.1% and 5.7% over $[0,4]$, and 5.0%, 1.2% and 9.2% over $[0,8]$. These numbers are in agreement with the estimated bias in Table 8.

GI	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
M	$[0,4]$	0	48.1 ± 1.2	$6.08 \times 10^{-4} \pm 4.59 \times 10^{-3}$	$-2.27 \times 10^{-4} \pm 4.19 \times 10^{-3}$
		1	45.8 ± 0.9	$6.77 \times 10^{-4} \pm 4.35 \times 10^{-3}$	$3.09 \times 10^{-5} \pm 3.77 \times 10^{-3}$
		2	45.3 ± 0.8	$6.73 \times 10^{-4} \pm 4.29 \times 10^{-3}$	$1.73 \times 10^{-4} \pm 3.70 \times 10^{-3}$
	$[0,8]$	0	48.3 ± 1.0	$1.64 \times 10^{-4} \pm 1.43 \times 10^{-3}$	$5.78 \times 10^{-4} \pm 9.58 \times 10^{-4}$
		1	45.8 ± 0.7	$1.31 \times 10^{-4} \pm 1.36 \times 10^{-3}$	$5.85 \times 10^{-4} \pm 8.45 \times 10^{-4}$
		2	45.3 ± 0.6	$1.16 \times 10^{-4} \pm 1.36 \times 10^{-3}$	$5.80 \times 10^{-4} \pm 8.28 \times 10^{-4}$
		Avg	46.4	3.95×10^{-4}	2.87×10^{-4}
E_4	$[0,4]$	0	47.8 ± 1.0	$4.56 \times 10^{-4} \pm 2.81 \times 10^{-3}$	$-4.30 \times 10^{-4} \pm 1.31 \times 10^{-3}$
		1	45.6 ± 0.8	$5.60 \times 10^{-4} \pm 2.67 \times 10^{-3}$	$-2.91 \times 10^{-4} \pm 1.20 \times 10^{-3}$
		2	45.2 ± 0.7	$5.60 \times 10^{-4} \pm 2.65 \times 10^{-3}$	$-2.19 \times 10^{-4} \pm 1.16 \times 10^{-3}$
	$[0,8]$	0	47.6 ± 0.8	$1.09 \times 10^{-4} \pm 8.67 \times 10^{-4}$	$1.53 \times 10^{-4} \pm 2.83 \times 10^{-4}$
		1	45.4 ± 0.6	$9.29 \times 10^{-5} \pm 8.32 \times 10^{-4}$	$1.57 \times 10^{-4} \pm 2.56 \times 10^{-4}$
		2	45.1 ± 0.5	$9.39 \times 10^{-5} \pm 8.27 \times 10^{-4}$	$1.61 \times 10^{-4} \pm 2.50 \times 10^{-4}$
		Avg	46.1	3.12×10^{-4}	-7.84×10^{-5}
H_2	$[0,4]$	0	46.1 ± 1.6	$2.20 \times 10^{-3} \pm 1.29 \times 10^{-2}$	$5.22 \times 10^{-2} \pm 6.85 \times 10^{-2}$
		1	43.9 ± 1.2	$1.92 \times 10^{-3} \pm 1.23 \times 10^{-2}$	$4.60 \times 10^{-2} \pm 6.07 \times 10^{-2}$
		2	43.6 ± 1.0	$1.80 \times 10^{-3} \pm 1.23 \times 10^{-2}$	$4.64 \times 10^{-2} \pm 5.96 \times 10^{-2}$
	$[0,8]$	0	47.0 ± 1.3	$2.22 \times 10^{-4} \pm 4.14 \times 10^{-3}$	$1.42 \times 10^{-2} \pm 1.47 \times 10^{-2}$
		1	44.5 ± 0.9	$1.52 \times 10^{-4} \pm 3.94 \times 10^{-3}$	$1.26 \times 10^{-2} \pm 1.29 \times 10^{-2}$
		2	44.1 ± 0.8	$1.78 \times 10^{-4} \pm 3.90 \times 10^{-3}$	$1.24 \times 10^{-2} \pm 1.26 \times 10^{-2}$
		Avg	44.9	1.08×10^{-3}	3.06×10^{-2}

Table 6: CONSTANT arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

GI	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
M	$[0,4]$	0	40.8 ± 1.1	$6.72 \times 10^{-2} \pm 3.06 \times 10^{-3}$	$3.91 \times 10^{-3} \pm 4.79 \times 10^{-3}$
		1	39.1 ± 0.8	$6.46 \times 10^{-2} \pm 2.94 \times 10^{-3}$	$3.69 \times 10^{-3} \pm 4.32 \times 10^{-3}$
		2	38.8 ± 0.8	$6.42 \times 10^{-2} \pm 2.91 \times 10^{-3}$	$3.72 \times 10^{-3} \pm 4.26 \times 10^{-3}$
	$[0,8]$	0	47.2 ± 0.9	$6.17 \times 10^{-2} \pm 1.11 \times 10^{-3}$	$5.35 \times 10^{-4} \pm 7.53 \times 10^{-4}$
		1	45.3 ± 0.7	$5.93 \times 10^{-2} \pm 9.63 \times 10^{-4}$	$5.69 \times 10^{-4} \pm 6.81 \times 10^{-4}$
		2	45.0 ± 0.6	$5.89 \times 10^{-2} \pm 9.16 \times 10^{-4}$	$5.80 \times 10^{-4} \pm 6.69 \times 10^{-4}$
		Avg	42.7	6.27×10^{-2}	2.17×10^{-3}
E_4	$[0,4]$	0	41.5 ± 0.8	$4.30 \times 10^{-2} \pm 2.02 \times 10^{-3}$	$1.03 \times 10^{-3} \pm 1.38 \times 10^{-3}$
		1	40.1 ± 0.7	$4.16 \times 10^{-2} \pm 1.95 \times 10^{-3}$	$1.02 \times 10^{-3} \pm 1.29 \times 10^{-3}$
		2	39.9 ± 0.6	$4.14 \times 10^{-2} \pm 1.94 \times 10^{-3}$	$1.03 \times 10^{-3} \pm 1.27 \times 10^{-3}$
	$[0,8]$	0	48.1 ± 0.7	$3.93 \times 10^{-2} \pm 6.11 \times 10^{-4}$	$1.95 \times 10^{-4} \pm 2.33 \times 10^{-4}$
		1	46.4 ± 0.6	$3.80 \times 10^{-2} \pm 5.70 \times 10^{-4}$	$1.98 \times 10^{-4} \pm 2.14 \times 10^{-4}$
		2	46.1 ± 0.5	$3.77 \times 10^{-2} \pm 5.54 \times 10^{-4}$	$2.00 \times 10^{-4} \pm 2.10 \times 10^{-4}$
		Avg	43.7	4.02×10^{-2}	6.12×10^{-4}
H_2	$[0,4]$	0	35.6 ± 1.2	$1.76 \times 10^{-1} \pm 9.31 \times 10^{-3}$	$-2.21 \times 10^{-1} \pm 6.04 \times 10^{-1}$
		1	33.8 ± 0.9	$1.67 \times 10^{-1} \pm 8.44 \times 10^{-3}$	$1.34 \times 10^{1} \pm 2.63 \times 10^{1}$
		2	33.4 ± 0.8	$1.65 \times 10^{-1} \pm 8.00 \times 10^{-3}$	$-6.00 \times 10^{-1} \pm 1.57 \times 10^{0}$
	$[0,8]$	0	41.9 ± 1.1	$1.65 \times 10^{-1} \pm 4.38 \times 10^{-3}$	$1.65 \times 10^{-2} \pm 1.34 \times 10^{-2}$
		1	39.9 ± 0.8	$1.57 \times 10^{-1} \pm 3.54 \times 10^{-3}$	$1.52 \times 10^{-2} \pm 1.16 \times 10^{-2}$
		2	39.4 ± 0.7	$1.55 \times 10^{-1} \pm 3.11 \times 10^{-3}$	$1.45 \times 10^{-2} \pm 1.10 \times 10^{-2}$
		Avg	37.3	1.64×10^{-1}	2.10×10^{0}

Table 7: LINEAR arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

GI	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
M	$[0,4]$	0	57.3 ± 1.1	$3.79 \times 10^{-2} \pm 3.45 \times 10^{-3}$	$-4.26 \times 10^{-3} \pm 4.49 \times 10^{-3}$
		1	55.2 ± 0.9	$3.66 \times 10^{-2} \pm 3.31 \times 10^{-3}$	$-3.78 \times 10^{-3} \pm 4.12 \times 10^{-3}$
		2	54.9 ± 0.8	$3.63 \times 10^{-2} \pm 3.27 \times 10^{-3}$	$-3.61 \times 10^{-3} \pm 4.06 \times 10^{-3}$
	$[0,8]$	0	60.2 ± 1.0	$3.86 \times 10^{-2} \pm 1.09 \times 10^{-3}$	$-6.34 \times 10^{-3} \pm 5.75 \times 10^{-4}$
		1	57.9 ± 0.8	$3.71 \times 10^{-2} \pm 1.00 \times 10^{-3}$	$-5.84 \times 10^{-3} \pm 5.16 \times 10^{-4}$
		2	57.4 ± 0.7	$3.68 \times 10^{-2} \pm 9.77 \times 10^{-4}$	$-5.73 \times 10^{-3} \pm 5.03 \times 10^{-4}$
		Avg	57.2	3.72×10^{-2}	-4.93×10^{-3}
E_4	$[0,4]$	0	57.7 ± 1.0	$2.39 \times 10^{-2} \pm 2.16 \times 10^{-3}$	$-1.47 \times 10^{-3} \pm 1.33 \times 10^{-3}$
		1	55.9 ± 0.8	$2.32 \times 10^{-2} \pm 2.10 \times 10^{-3}$	$-1.29 \times 10^{-3} \pm 1.25 \times 10^{-3}$
		2	55.6 ± 0.8	$2.30 \times 10^{-2} \pm 2.08 \times 10^{-3}$	$-1.25 \times 10^{-3} \pm 1.23 \times 10^{-3}$
	$[0,8]$	0	60.1 ± 0.9	$2.41 \times 10^{-2} \pm 6.94 \times 10^{-4}$	$-1.97 \times 10^{-3} \pm 1.89 \times 10^{-4}$
		1	58.0 ± 0.7	$2.33 \times 10^{-2} \pm 6.40 \times 10^{-4}$	$-1.82 \times 10^{-3} \pm 1.70 \times 10^{-4}$
		2	57.6 ± 0.6	$2.31 \times 10^{-2} \pm 6.26 \times 10^{-4}$	$-1.80 \times 10^{-3} \pm 1.67 \times 10^{-4}$
		Avg	57.5	2.34×10^{-2}	-1.60×10^{-3}
H_2	$[0,4]$	0	51.7 ± 1.5	$1.02 \times 10^{-1} \pm 9.16 \times 10^{-3}$	$8.77 \times 10^{-3} \pm 8.38 \times 10^{-2}$
		1	48.6 ± 1.0	$9.56 \times 10^{-2} \pm 8.47 \times 10^{-3}$	$1.21 \times 10^{-2} \pm 7.23 \times 10^{-2}$
		2	48.2 ± 0.9	$9.49 \times 10^{-2} \pm 8.35 \times 10^{-3}$	$1.39 \times 10^{-2} \pm 7.12 \times 10^{-2}$
	$[0,8]$	0	55.6 ± 1.5	$1.07 \times 10^{-1} \pm 3.59 \times 10^{-3}$	$-8.65 \times 10^{-2} \pm 9.21 \times 10^{-3}$
		1	52.6 ± 1.1	$1.01 \times 10^{-1} \pm 2.97 \times 10^{-3}$	$-7.64 \times 10^{-2} \pm 7.65 \times 10^{-3}$
		2	52.1 ± 0.9	$9.99 \times 10^{-2} \pm 2.80 \times 10^{-3}$	$-7.44 \times 10^{-2} \pm 7.13 \times 10^{-3}$
		Avg	51.5	1.00×10^{-1}	-3.38×10^{-2}

Table 8: QUADRATIC arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

We now estimate the bias reduction achieved by our estimators by two performance measures. The first performance measure computes the absolute difference between (i) the average of the estimate of interest over the 100 replications and (ii) the average of the direct estimate $\bar{W}(t)$ over the same 100 replications. Tables 9, 10 and 11 illustrate these results when constant, linear and quadratic arrival rates are used, respectively.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	0.3	0.2	0.2	0.6	0.3	0.3	0.5	0.0	0.0
		1	0.1	0.2	0.2	0.3	0.1	0.1	0.3	0.2	0.2
		2	0.0	0.1	0.1	0.3	0.0	0.3	0.2	0.2	0.2
	$[0,8]$	0	0.9	0.2	0.2	0.7	0.9	0.9	0.5	0.8	0.8
		1	0.5	0.1	0.1	0.4	0.5	0.4	0.2	0.4	0.2
		2	0.4	0.1	0.1	0.3	0.4	0.3	0.1	0.3	0.1
		Avg	0.4	0.2	0.2	0.4	0.4	0.4	0.3	0.3	0.2
E_4	$[0,4]$	0	0.6	0.1	0.2	0.8	0.6	0.6	0.7	0.4	0.4
		1	0.4	0.1	0.1	0.6	0.4	0.4	0.5	0.2	0.2
		2	0.3	0.1	0.0	0.5	0.3	0.3	0.5	0.2	0.2
	$[0,8]$	0	0.6	0.1	0.2	0.4	0.6	0.6	0.3	0.5	0.5
		1	0.2	0.1	0.0	0.0	0.2	0.2	0.1	0.1	0.1
		2	0.2	0.1	0.0	0.0	0.2	0.2	0.1	0.1	0.1
		Avg	0.4	0.1	0.1	0.4	0.4	0.4	0.4	0.2	0.2
H_2	$[0,4]$	0	4.9	3.3	0.1	3.6	4.8	4.8	86.8	10.4	10.4
		1	4.9	3.7	1.3	3.8	4.8	4.8	74.2	9.5	9.5
		2	5.2	4.2	2.0	4.2	5.2	5.2	74.2	10.0	10.0
	$[0,8]$	0	3.7	2.0	1.6	3.4	3.7	3.7	3.1	4.7	4.7
		1	3.4	2.4	0.4	3.2	3.4	3.4	3.0	4.1	4.1
		2	3.4	2.6	0.9	3.2	3.4	3.4	3.1	4.1	4.1
		Avg	4.3	3.0	1.0	3.6	4.2	4.2	40.7	7.1	7.1

Table 9: CONSTANT arrival rate: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

Another performance measure of interest is the absolute relative error of the estimate in each sample path. We average this relative error over 100 replications. Tables 12, 13 and 14 report the results. These results measure the ability of the estimator to match the direct estimator $\bar{W}(t)$ over each sample path, and thus strongly favor the sample-path based estimator $\bar{W}_{L,\lambda,r,\gamma}(t)$. Indeed, that is the only estimator that provides significant improvement over the indirect estimator $\bar{W}_{L,\lambda}(t)$ from this perspective.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	5.9	0.8	0.8	2.0	0.7	0.7	2.3	0.7	0.7
		1	6.3	0.3	0.3	0.8	0.2	0.8	1.2	0.2	1.2
		2	6.3	0.4	0.4	0.7	0.3	0.7	1.1	0.3	1.1
	$[0,8]$	0	6.8	0.1	0.1	0.1	0.7	0.7	0.3	0.6	0.6
		1	6.6	0.5	0.5	0.3	1.0	1.0	0.1	0.9	0.1
		2	6.5	0.4	0.4	0.3	1.0	1.0	0.1	0.8	0.1
		Avg	6.4	0.4	0.4	0.7	0.7	0.8	0.8	0.6	0.6
E_4	$[0,4]$	0	4.2	3.0	0.3	0.6	0.1	0.1	0.9	0.2	0.9
		1	4.7	2.9	0.0	0.2	0.7	0.2	0.1	0.5	0.1
		2	4.7	2.8	0.0	0.2	0.7	0.2	0.1	0.5	0.1
	$[0,8]$	0	4.7	2.2	0.4	0.5	0.8	0.8	0.4	0.7	0.7
		1	4.5	2.1	0.3	0.6	0.8	0.8	0.4	0.7	0.7
		2	4.4	2.2	0.3	0.5	0.7	0.7	0.4	0.6	0.6
		Avg	4.5	2.5	0.2	0.4	0.6	0.5	0.4	0.5	0.5
H_2	$[0,4]$	0	18.7	13.2	2.4	11.6	3.4	3.4	69.3	18.9	18.9
		1	19.4	13.8	2.5	5.2	5.7	5.7	43.9	1233.8	43.9
		2	19.6	14.0	2.7	4.5	6.3	6.3	47.8	44.6	44.6
	$[0,8]$	0	19.0	13.2	1.6	5.3	4.5	4.5	21.1	5.4	5.4
		1	19.0	13.7	3.0	1.7	5.9	5.9	16.8	6.7	6.7
		2	19.1	13.8	3.2	0.7	6.3	6.3	4.2	7.0	7.0
		Avg	19.1	13.6	2.6	4.8	5.4	5.4	33.9	219.4	21.1

Table 10: LINEAR arrival rate: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	4.5	0.4	0.4	0.1	0.7	0.7	0.7	1.4	1.4
		1	4.0	0.3	0.3	0.1	0.4	0.4	0.4	1.0	1.0
		2	3.8	0.4	0.4	0.2	0.3	0.3	0.3	0.9	0.9
	$[0,8]$	0	3.1	0.4	0.4	1.3	0.9	0.9	0.8	0.9	0.9
		1	2.7	0.5	0.5	1.4	1.0	1.0	0.6	0.7	0.7
		2	2.5	0.4	0.4	1.5	1.1	1.1	0.4	0.6	0.6
		Avg	3.5	0.4	0.4	0.8	0.7	0.7	0.5	0.9	0.9
E_4	$[0,4]$	0	2.6	1.2	0.2	0.2	0.1	0.1	0.3	0.7	0.7
		1	2.0	1.0	0.1	0.5	0.3	0.3	0.1	0.2	0.2
		2	1.9	1.0	0.1	0.7	0.4	0.4	0.2	0.1	0.2
	$[0,8]$	0	1.7	0.7	0.2	1.0	0.8	0.8	0.5	0.6	0.6
		1	1.3	0.6	0.1	1.2	1.0	1.0	0.2	0.3	0.3
		2	1.2	0.6	0.1	1.2	1.1	1.1	0.1	0.2	0.2
		Avg	1.8	0.9	0.1	0.8	0.6	0.6	0.2	0.3	0.4
H_2	$[0,4]$	0	17.3	13.9	7.1	3.9	7.9	7.9	135.2	10.2	10.2
		1	17.0	13.6	6.7	5.8	8.8	8.8	117.4	11.5	11.5
		2	16.5	13.0	6.0	5.5	8.5	8.5	116.6	11.3	11.3
	$[0,8]$	0	13.3	10.2	4.0	0.4	3.0	3.0	55.1	1.0	1.0
		1	13.0	10.1	4.4	1.1	3.8	3.8	10.2	2.5	2.5
		2	12.7	10.0	4.5	1.1	3.7	3.7	6.1	2.6	2.6
		Avg	15.0	11.8	5.5	3.0	6.0	6.0	73.4	6.5	6.5

Table 11: QUADRATIC arrival rate: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	5.6 ± 0.7	4.3 ± 0.7	4.3 ± 0.7	5.1 ± 0.7	4.9 ± 0.7	5.2 ± 0.7	5.2 ± 0.7
		1	5.3 ± 0.7	4.4 ± 0.6	4.4 ± 0.6	4.9 ± 0.7	4.8 ± 0.7	5.1 ± 0.7	5.1 ± 0.7
		2	5.2 ± 0.7	4.4 ± 0.6	4.4 ± 0.6	5.0 ± 0.7	4.8 ± 0.7	5.2 ± 0.7	5.1 ± 0.7
	$[0,8]$	0	3.1 ± 0.5	2.2 ± 0.3	2.2 ± 0.3	3.3 ± 0.5	3.0 ± 0.5	3.1 ± 0.5	2.8 ± 0.4
		1	2.9 ± 0.4	2.2 ± 0.3	2.2 ± 0.3	3.0 ± 0.5	2.8 ± 0.4	2.9 ± 0.5	2.7 ± 0.4
		2	2.7 ± 0.4	2.2 ± 0.3	2.2 ± 0.3	3.0 ± 0.5	2.7 ± 0.4	2.9 ± 0.5	2.6 ± 0.4
		Avg	4.1	3.3	3.3	4.1	3.8	4.1	3.9
E_4	$[0,4]$	0	4.5 ± 0.6	1.9 ± 0.3	2.0 ± 0.3	3.6 ± 0.5	3.7 ± 0.6	2.9 ± 0.5	3.1 ± 0.5
		1	3.7 ± 0.5	2.3 ± 0.3	1.7 ± 0.3	3.3 ± 0.5	3.3 ± 0.5	2.7 ± 0.4	2.6 ± 0.4
		2	3.6 ± 0.5	2.5 ± 0.4	1.8 ± 0.3	3.3 ± 0.5	3.2 ± 0.5	2.8 ± 0.4	2.6 ± 0.4
	$[0,8]$	0	2.2 ± 0.3	1.1 ± 0.2	1.1 ± 0.2	2.3 ± 0.3	2.0 ± 0.3	2.0 ± 0.3	1.7 ± 0.3
		1	1.7 ± 0.3	1.3 ± 0.2	1.1 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.9 ± 0.3	1.4 ± 0.2
		2	1.7 ± 0.2	1.4 ± 0.2	1.0 ± 0.2	2.0 ± 0.3	1.6 ± 0.2	1.9 ± 0.3	1.4 ± 0.2
		Avg	2.9	1.7	1.5	2.8	2.6	2.4	2.2
H_2	$[0,4]$	0	12.6 ± 1.7	11.2 ± 1.6	13.1 ± 2.1	12.5 ± 1.7	12.6 ± 1.8	83.7 ± 24.1	21.1 ± 3.4
		1	12.9 ± 1.9	11.9 ± 1.7	12.9 ± 2.0	13.0 ± 1.9	13.1 ± 1.9	77.3 ± 23.6	19.1 ± 3.1
		2	12.9 ± 1.8	11.7 ± 1.7	12.6 ± 1.9	12.7 ± 1.9	13.0 ± 1.9	77.5 ± 23.6	18.7 ± 3.0
	$[0,8]$	0	9.4 ± 1.3	7.9 ± 1.1	8.2 ± 1.2	9.4 ± 1.2	9.5 ± 1.3	10.0 ± 1.4	9.9 ± 1.4
		1	9.8 ± 1.3	8.7 ± 1.1	7.8 ± 1.2	9.8 ± 1.2	9.8 ± 1.3	10.2 ± 1.3	10.1 ± 1.3
		2	9.7 ± 1.3	8.7 ± 1.1	8.0 ± 1.1	9.7 ± 1.2	9.7 ± 1.2	10.0 ± 1.3	10.0 ± 1.3
		Avg	11.2	10.0	10.4	11.2	11.3	44.8	14.8

Table 12: CONSTANT arrival rate: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	7.0 ± 1.1	4.3 ± 0.6	4.3 ± 0.6	6.6 ± 0.9	6.3 ± 0.8	6.2 ± 0.9	5.8 ± 0.8
		1	7.2 ± 1.1	4.4 ± 0.6	4.4 ± 0.6	6.1 ± 0.9	5.8 ± 0.8	5.7 ± 0.8	5.3 ± 0.7
		2	7.1 ± 1.1	4.4 ± 0.6	4.4 ± 0.6	6.0 ± 0.8	5.7 ± 0.7	5.6 ± 0.8	5.2 ± 0.7
	$[0,8]$	0	6.6 ± 0.7	2.0 ± 0.3	2.0 ± 0.3	3.4 ± 0.6	3.3 ± 0.6	3.0 ± 0.5	3.1 ± 0.5
		1	6.6 ± 0.7	1.9 ± 0.3	1.9 ± 0.3	3.1 ± 0.5	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
		2	6.5 ± 0.6	1.9 ± 0.3	1.9 ± 0.3	3.0 ± 0.4	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
		Avg	6.8	3.2	3.2	4.7	4.5	4.3	4.1
E_4	$[0,4]$	0	5.2 ± 0.6	3.2 ± 0.5	2.3 ± 0.4	3.9 ± 0.5	3.6 ± 0.6	3.2 ± 0.5	3.0 ± 0.5
		1	5.3 ± 0.7	3.2 ± 0.4	2.2 ± 0.3	3.7 ± 0.5	3.7 ± 0.5	2.9 ± 0.5	3.0 ± 0.4
		2	5.2 ± 0.6	3.1 ± 0.5	2.2 ± 0.3	3.6 ± 0.5	3.6 ± 0.5	2.9 ± 0.5	2.9 ± 0.4
	$[0,8]$	0	4.5 ± 0.5	2.1 ± 0.2	1.2 ± 0.2	2.4 ± 0.4	2.2 ± 0.3	2.0 ± 0.3	1.9 ± 0.3
		1	4.4 ± 0.4	2.2 ± 0.2	1.1 ± 0.2	2.2 ± 0.4	1.9 ± 0.3	1.8 ± 0.3	1.6 ± 0.3
		2	4.4 ± 0.4	2.3 ± 0.2	1.1 ± 0.2	2.1 ± 0.3	1.7 ± 0.3	1.8 ± 0.3	1.6 ± 0.2
		Avg	4.8	2.7	1.7	3.0	2.8	2.4	2.3
H_2	$[0,4]$	0	19.6 ± 2.1	15.4 ± 2.0	14.6 ± 2.2	20.7 ± 4.6	14.2 ± 2.4	198.9 ± 120.2	55.5 ± 34.2
		1	20.2 ± 2.2	16.0 ± 2.0	14.4 ± 2.0	16.9 ± 3.2	14.2 ± 2.4	184.8 ± 122.2	759.9 ± 1408.2
		2	20.3 ± 2.2	16.0 ± 2.1	14.3 ± 1.9	16.6 ± 3.3	13.7 ± 2.3	188.7 ± 122.4	90.1 ± 119.7
	$[0,8]$	0	17.3 ± 1.8	12.5 ± 1.5	8.2 ± 1.0	12.0 ± 2.1	9.8 ± 1.3	31.6 ± 20.0	10.2 ± 1.4
		1	18.0 ± 1.7	13.2 ± 1.6	8.0 ± 1.0	9.8 ± 1.7	9.8 ± 1.3	24.3 ± 16.5	10.1 ± 1.4
		2	18.2 ± 1.7	13.3 ± 1.6	8.0 ± 1.0	9.4 ± 1.4	9.9 ± 1.3	14.1 ± 8.8	10.2 ± 1.4
		Avg	18.9	14.4	11.2	14.2	11.9	107.1	156.0

Table 13: LINEAR arrival rate: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	5.8 ± 0.7	3.2 ± 0.5	3.2 ± 0.5	4.8 ± 0.7	4.7 ± 0.6	4.6 ± 0.6	4.5 ± 0.6
		1	5.6 ± 0.7	3.2 ± 0.5	3.2 ± 0.5	5.1 ± 0.8	4.6 ± 0.7	4.7 ± 0.7	4.3 ± 0.6
		2	5.5 ± 0.7	3.1 ± 0.5	3.1 ± 0.5	5.1 ± 0.8	4.6 ± 0.7	4.7 ± 0.7	4.3 ± 0.6
	$[0,8]$	0	3.5 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.9 ± 0.4	2.5 ± 0.4	2.6 ± 0.3	2.5 ± 0.3
		1	3.2 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.9 ± 0.4	2.4 ± 0.4	2.4 ± 0.3	2.3 ± 0.3
		2	3.1 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.8 ± 0.4	2.4 ± 0.4	2.3 ± 0.3	2.2 ± 0.3
		Avg	4.5	2.4	2.4	3.9	3.5	3.6	3.4
E_4	$[0,4]$	0	3.5 ± 0.5	2.2 ± 0.3	1.8 ± 0.3	3.2 ± 0.4	2.8 ± 0.4	2.6 ± 0.4	2.4 ± 0.3
		1	2.9 ± 0.5	2.2 ± 0.3	1.6 ± 0.2	3.0 ± 0.4	2.5 ± 0.4	2.4 ± 0.4	2.0 ± 0.3
		2	2.8 ± 0.4	2.2 ± 0.3	1.6 ± 0.2	3.1 ± 0.4	2.5 ± 0.3	2.5 ± 0.4	2.0 ± 0.3
	$[0,8]$	0	1.9 ± 0.3	1.1 ± 0.2	0.8 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
		1	1.6 ± 0.3	1.1 ± 0.2	0.9 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
		2	1.6 ± 0.2	1.1 ± 0.2	0.9 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
		Avg	2.4	1.7	1.3	2.6	2.1	2.0	1.7
H_2	$[0,4]$	0	16.9 ± 2.2	14.0 ± 2.0	14.2 ± 1.9	13.7 ± 2.0	13.8 ± 2.0	128.6 ± 28.2	23.0 ± 4.9
		1	17.2 ± 2.2	14.4 ± 2.0	12.3 ± 1.8	13.3 ± 2.0	14.1 ± 1.9	118.4 ± 28.6	20.6 ± 4.3
		2	17.2 ± 2.2	14.4 ± 2.0	11.7 ± 1.8	13.3 ± 2.0	14.1 ± 2.0	119.2 ± 28.9	20.4 ± 4.3
	$[0,8]$	0	12.4 ± 1.5	9.9 ± 1.3	7.6 ± 1.0	8.5 ± 1.2	8.2 ± 1.1	56.8 ± 24.0	8.6 ± 1.1
		1	12.6 ± 1.4	10.2 ± 1.2	7.1 ± 1.0	8.0 ± 1.1	8.0 ± 1.1	19.7 ± 11.5	8.2 ± 1.1
		2	12.6 ± 1.4	10.3 ± 1.3	7.0 ± 1.0	8.2 ± 1.1	8.1 ± 1.2	16.6 ± 9.7	8.3 ± 1.1
		Avg	14.8	12.2	10.0	10.8	11.0	76.6	14.8

Table 14: QUADRATIC arrival rate: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

2.4 A Closer Look at $\bar{W}_{L,\lambda,q,p}(t)$ for H_2 Service

We have included H_2 service in order to illustrate difficult cases; we expect to see *greater* bias in the indirect estimator $\bar{W}_{L,\lambda}(t)$ and *larger* confidence intervals due to the greater variability in H_2 compared to other service times. However, the performance of the quadratic estimators in Section 2.3 is exceptionally poor. Since much of the problem with $\bar{W}_{L,\lambda,q}(t)$ is caused by dividing by small numbers and there is clear improvement in going from $\bar{W}_{L,\lambda,q}(t)$ to $\bar{W}_{L,\lambda,q,p}(t)$, we focus on better understanding the performance of $\bar{W}_{L,\lambda,q,p}(t)$ in this section.

Figure 56-73 show the histograms of $W_{L,\lambda,q,p}(t)$ from the 100 replications for different arrival rate functions, QoS parameter β values and estimate intervals ($[0,4]$ and $[0,8]$). When the arrival rate function is linear and the estimate interval is $[0,4]$ (Figures 62, 64 and 66), we observe $W_{L,\lambda,q,p}(t)$ values that are less than 0 or greater than 2. We have used the same arrival process and service times for different values of β, and observe that these extreme $W_{L,\lambda,q,p}(t)$ values for different values of β are caused by the same set of six sample paths. Figures 74-79 illustrate the plots of $L(t)$ (Here we use $\beta = 0$. The plots look similar for other values of β) for each of the sample paths. We observe that $L(0)$ and/or $L(4)$ values are too low or too high than expected, making corrections to the estimators via $\bar{W}_{L,\lambda,q,p}(t)$ invalid. We further find that these unusual $L(0)$ and/or $L(4)$ values are caused by greater variability in service times (i.e., too many small and/or too many large service times) than expected from the H_2 distribution; Figures 80-85 show the histogram of services times in $[0,4]$ for each of the sample paths. For instance, Figure 80 shows that the average service time is 1.8 (whereas it is generated from H_2 distribution with mean 1), with maximum value 35.5 and 10 values that are greater than or equal to 10. We also see an extreme $W_{L,\lambda,q,p}(t)$ value for each value of β on $[0,4]$ when the arrival rate function is quadratic (Figures 68, 70 and 72). This outlier is from the same sample path, whose $L(t)$ plot and histogram of service times are given in Figures 86-87.

Table 15 shows the effect of removing the outliers. We let $\bar{W}_{L,\lambda,q,p'}(t)$ be the new waiting time estimate after removing the (at most six) outliers. $\#O$ indicates the number of outliers in each of the 100 replications (We also note that there was no $W_{L,\lambda,q,p}(t)$ value less than 0 or greater than 2 in the models with M and E_4 service times). The halfwidths of the confidence intervals for linear arrival rate on the interval $[0,4]$ are reduced dramatically. Consequently, Table 16 and Table 17 show that the absolute difference of the estimates from the direct estimate and the average of the absolute relative error of the estimate from the direct estimate in each sample path are dramatically reduced.

		Constant			Linear			Quadratic		
Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{(L,\lambda,q,p)'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{(L,\lambda,q,p)'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$
[0, 4]	0	0.965 ± 0.063	0	0.965 ± 0.063	1.230 ± 0.568	6	0.902 ± 0.066	0.985 ± 0.072	1	1.006 ± 0.060
	1	0.927 ± 0.049	0	0.927 ± 0.050	-11.332 ± 24.224	5	0.863 ± 0.049	0.916 ± 0.053	1	0.932 ± 0.044
	2	0.919 ± 0.046	0	0.919 ± 0.047	1.444 ± 1.305	4	0.866 ± 0.050	0.907 ± 0.051	1	0.922 ± 0.042
[0, 8]	0	1.031 ± 0.026	0	1.031 ± 0.026	1.009 ± 0.026	0	1.009 ± 0.027	1.078 ± 0.031	0	1.078 ± 0.032
	1	0.980 ± 0.019	0	0.980 ± 0.019	0.954 ± 0.019	0	0.954 ± 0.019	1.007 ± 0.022	0	1.007 ± 0.022
	2	0.972 ± 0.017	0	0.972 ± 0.017	0.942 ± 0.015	0	0.942 ± 0.016	0.995 ± 0.019	0	0.995 ± 0.019

Table 15: Constant, linear and quadratic arrival rate functions with H_2 service: Waiting time estimates by $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ with associated 95% confidence intervals. $\bar{W}_{L,\lambda,q,p'}(t)$ is the new waiting time estimate after removing the outliers. The number of outliers removed in each case is given under $\#O$.

		Constant		Linear		Quadratic	
Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
[0, 4]	0	10.4	10.4	18.9	13.9	10.2	8.2
	1	9.5	9.5	1233.8	14.3	11.5	9.9
	2	10.0	10.0	44.6	13.2	11.3	9.7
[0, 8]	0	4.7	4.7	5.4	5.4	1.0	1.0
	1	4.1	4.1	6.7	6.7	2.5	2.5
	2	4.1	4.1	7.0	7.0	2.6	2.6
	Avg	7.1	7.1	219.4	10.1	6.5	5.7

Table 16: Constant, linear and quadratic arrival rate functions with H_2 service: absolute difference of the waiting time estimates $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

		Constant		Linear		Quadratic	
Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
[0, 4]	0	21.1 ± 3.4	21.1 ± 3.4	55.5 ± 34.2	25.7 ± 4.6	23.0 ± 4.9	21.3 ± 3.6
	1	19.1 ± 3.1	19.1 ± 3.1	759.9 ± 1408.2	23.2 ± 3.9	20.6 ± 4.3	19.2 ± 3.3
	2	18.7 ± 3.0	18.7 ± 3.1	90.1 ± 119.7	23.2 ± 4.0	20.4 ± 4.3	19.0 ± 3.3
[0, 8]	0	9.9 ± 1.4	9.9 ± 1.4	10.2 ± 1.4	10.2 ± 1.4	8.6 ± 1.1	8.6 ± 1.1
	1	10.1 ± 1.3	10.1 ± 1.4	10.1 ± 1.4	10.1 ± 1.4	8.2 ± 1.1	8.2 ± 1.1
	2	10.0 ± 1.3	10.0 ± 1.3	10.2 ± 1.4	10.2 ± 1.4	8.3 ± 1.1	8.3 ± 1.2
	Avg	14.8	14.8	156.0	17.1	14.9	14.1

Table 17: Constant, linear and quadratic arrival rate functions with H_2 service: average of the absolute relative error of the waiting time estimates $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

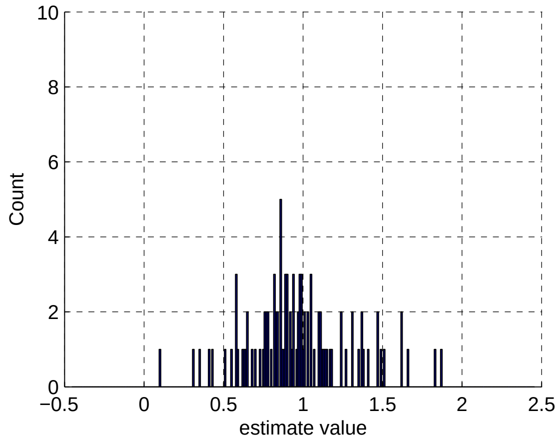

Figure 56: QoS $\beta = 0$: Estimates on $[0, 4]$

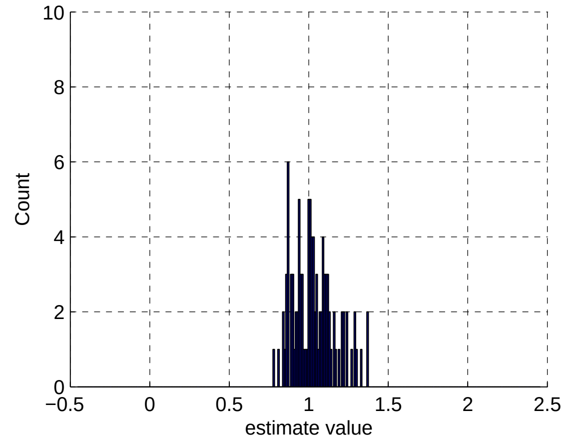

Figure 57: QoS $\beta = 0$: Estimates on $[0, 8]$

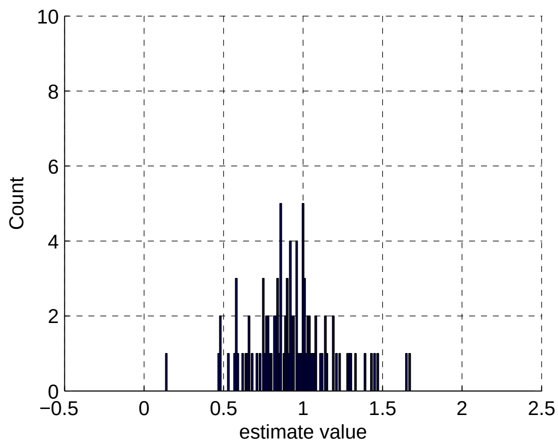

Figure 58: QoS $\beta = 1$: Estimates on $[0, 4]$

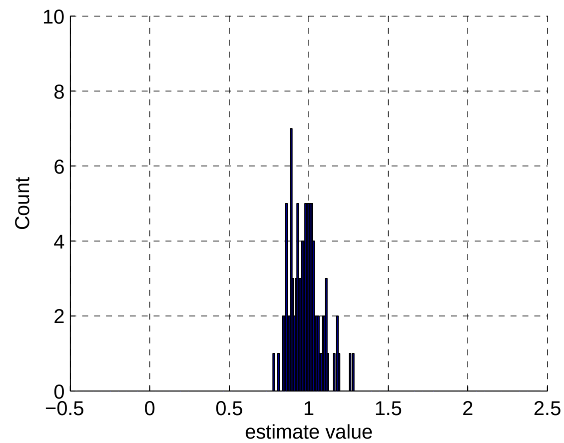

Figure 59: QoS $\beta = 1$: Estimates on $[0, 8]$

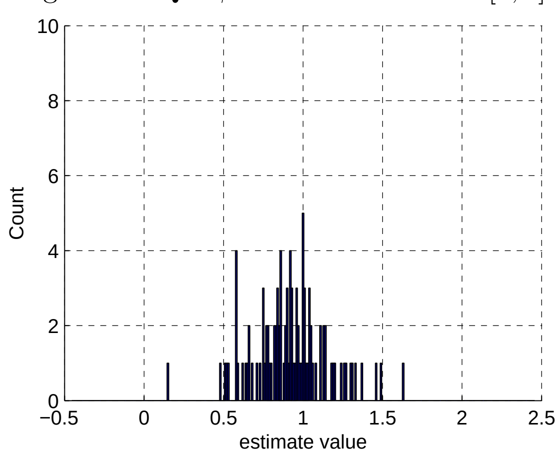

Figure 60: QoS $\beta = 2$: Estimates on $[0, 4]$

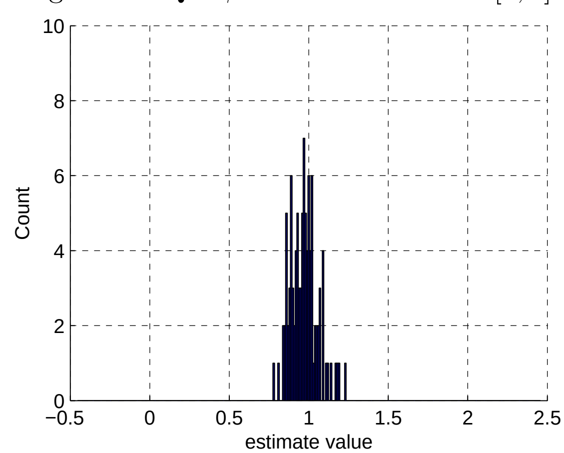

Figure 61: QoS $\beta = 2$: Estimates on $[0, 8]$

Figures 56-61: Constant arrival rate and H_2 service: Histograms (bin size: 0.01) of $W_{L,\lambda,q,p}(t)$ over 100 replications.

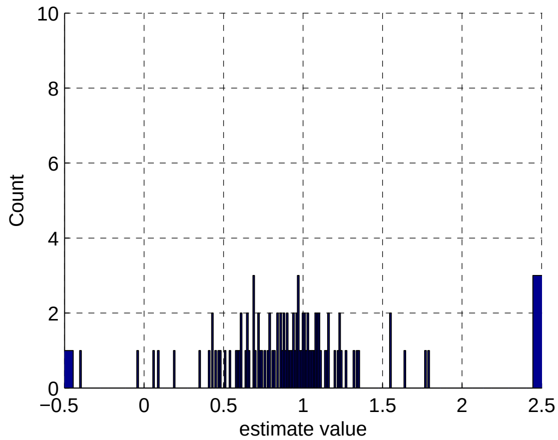

Figure 62: QoS $\beta = 0$: Estimates on $[0, 4]$

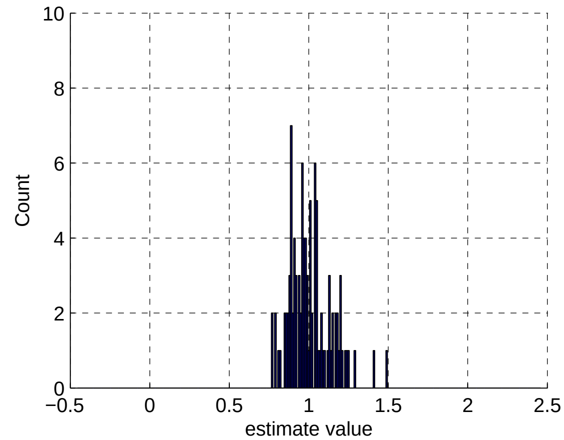

Figure 63: QoS $\beta = 0$: Estimates on $[0, 8]$

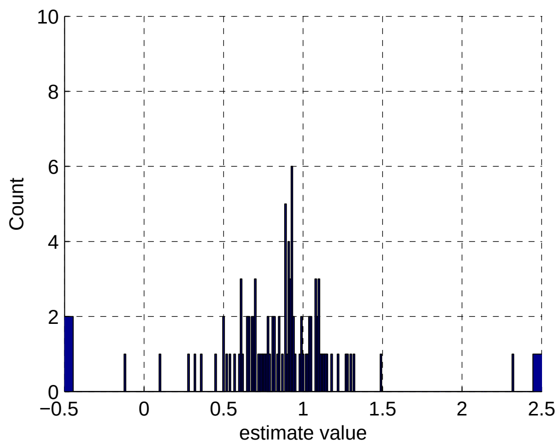

Figure 64: QoS $\beta = 1$: Estimates on $[0, 4]$

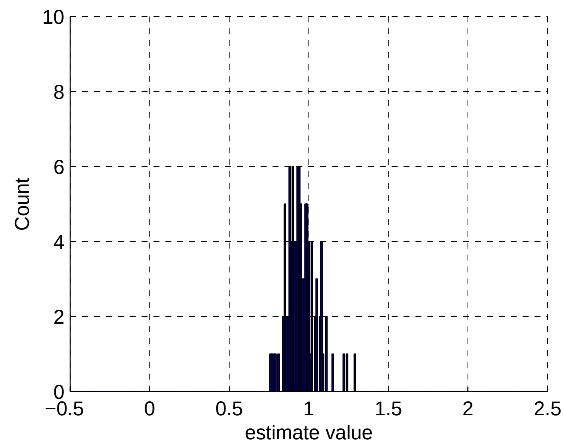

Figure 65: QoS $\beta = 1$: Estimates on $[0, 8]$

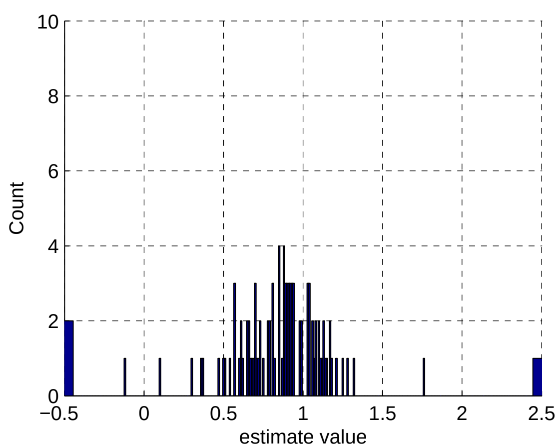

Figure 66: QoS $\beta = 2$: Estimates on $[0, 4]$

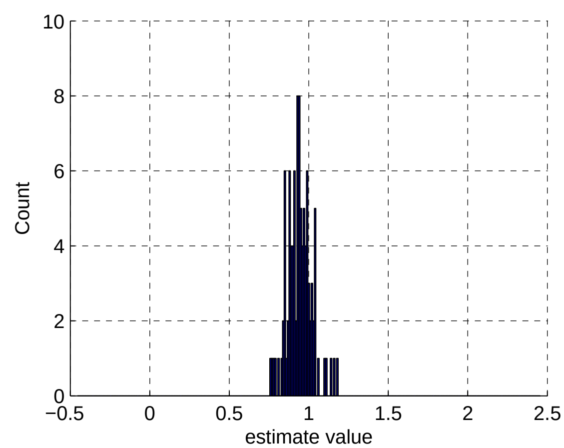

Figure 67: QoS $\beta = 2$: Estimates on $[0, 8]$

Figures 62-67: Linear arrival rate and H_2 service: Histograms (bin size: 0.01) of $W_{L,\lambda,q,p}(t)$ over 100 replications. Counts at -0.5 (2.5) indicate # estimates that are less (greater) than -0.5 (2.5)

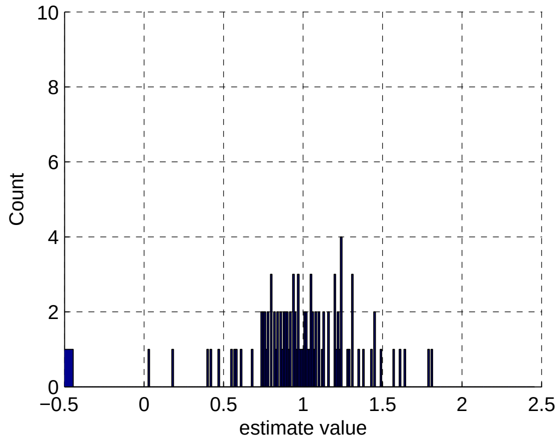

Figure 68: QoS $\beta = 0$: Estimates on $[0, 4]$

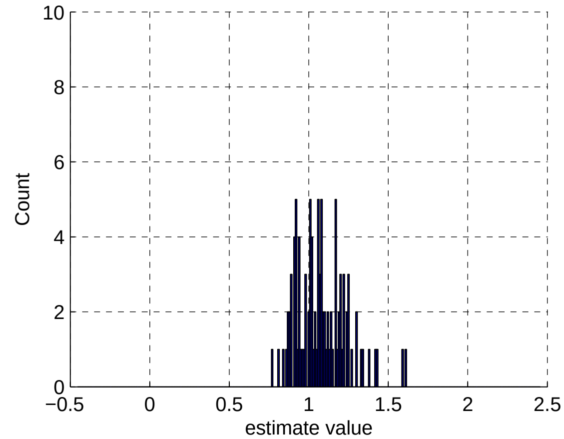

Figure 69: QoS $\beta = 0$: Estimates on $[0, 8]$

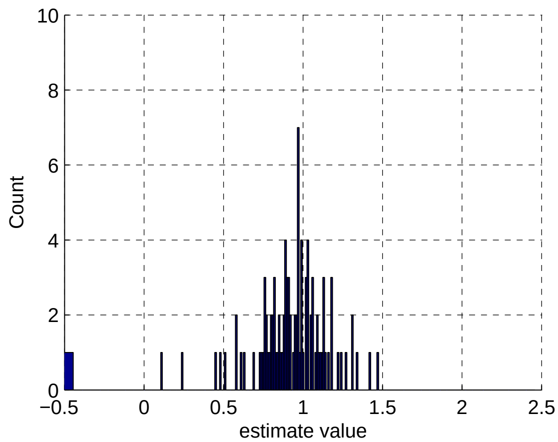

Figure 70: QoS $\beta = 1$: Estimates on $[0, 4]$

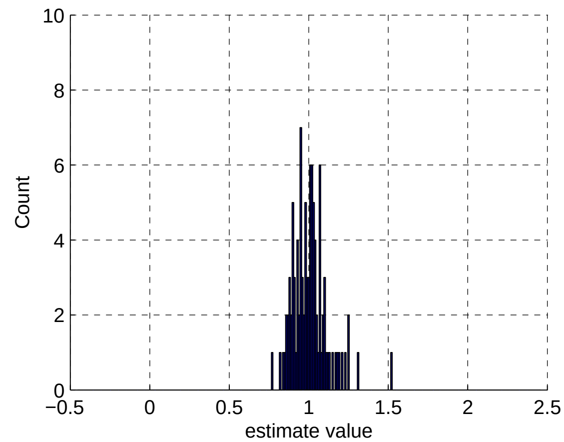

Figure 71: QoS $\beta = 1$: Estimates on $[0, 8]$

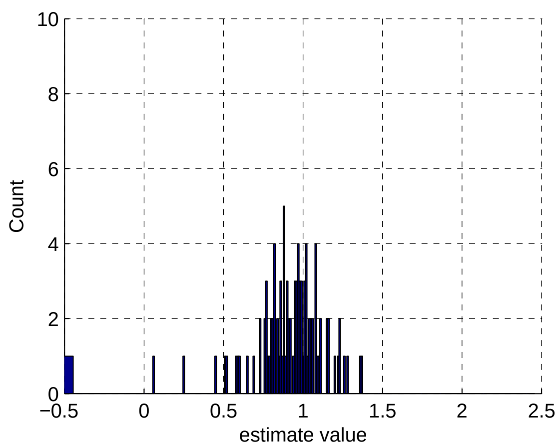

Figure 72: QoS $\beta = 2$: Estimates on $[0, 4]$.

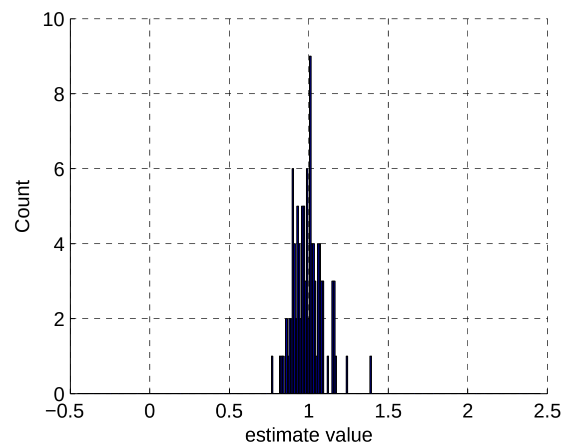

Figure 73: QoS $\beta = 2$: Estimates on $[0, 8]$

Figures 68-73: Quadratic arrival rate and H_2 service: Histograms (bin size: 0.01) of $W_{L,\lambda,q,p}(t)$ over 100 replications. Counts at -0.5 indicate # estimates that are less than -0.5.

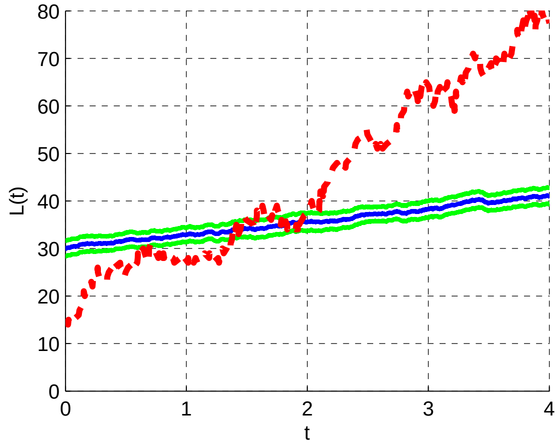

Figure 74: Outlier 1

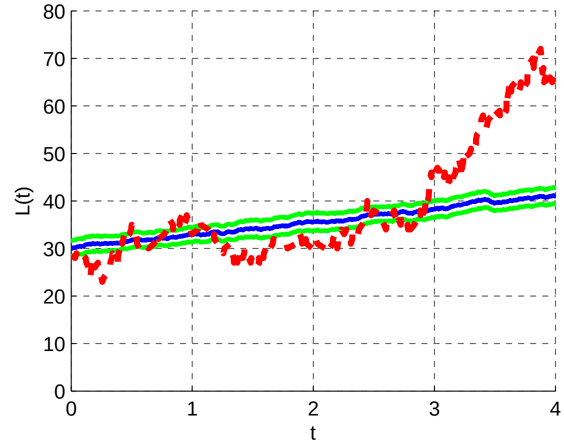

Figure 75: Outlier 2

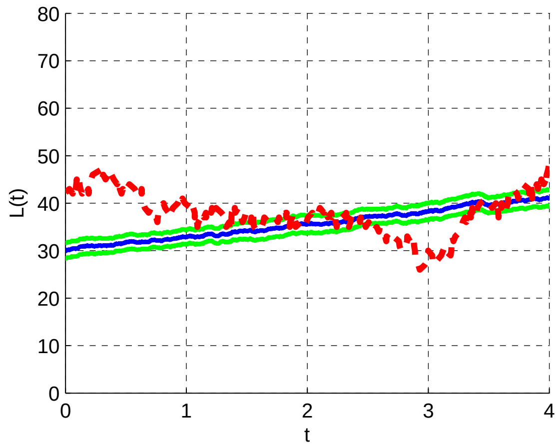

Figure 76: Outlier 3

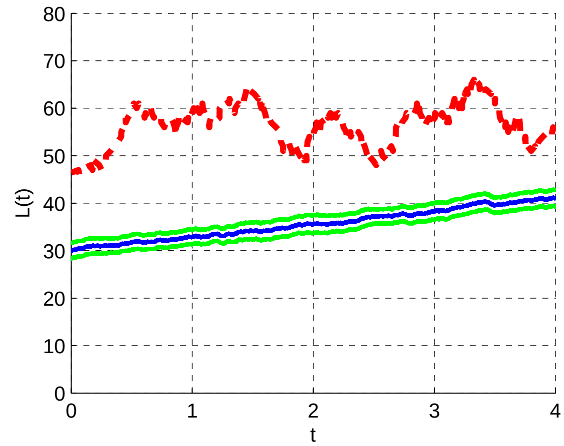

Figure 77: Outlier 4

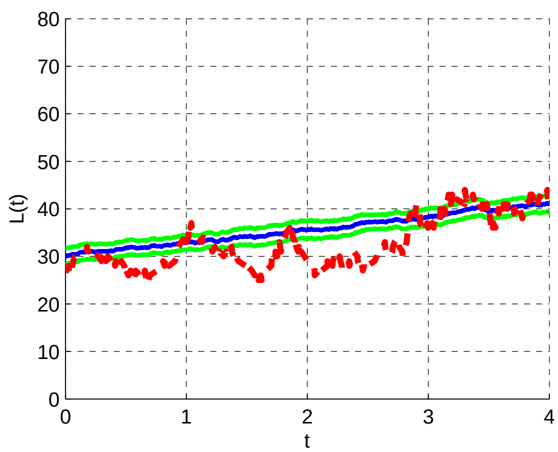

Figure 78: Outlier 5

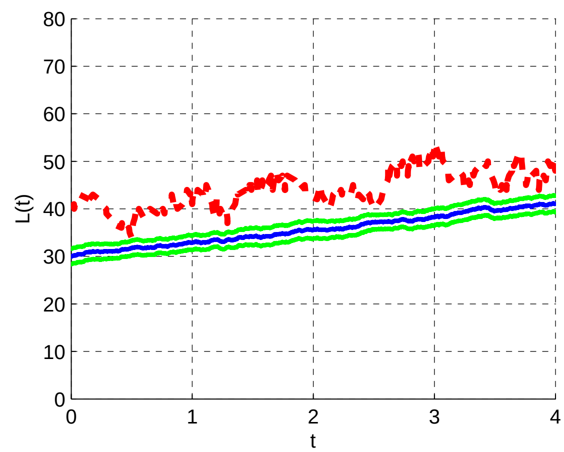

Figure 79: Outlier 6

Figures 74-79: Linear arrival rate and H_2 service: $L(t)$ (dashed line) of sample paths with extreme values of $W_{L,\lambda,q,p}(t)$ ($W_{L,\lambda,q,p}(t) < 0$ or $W_{L,\lambda,q,p}(t) > 2$) as well as average $L(t)$ with associated 95% confidence interval based on 100 replications. $\beta = 0$ in all cases.

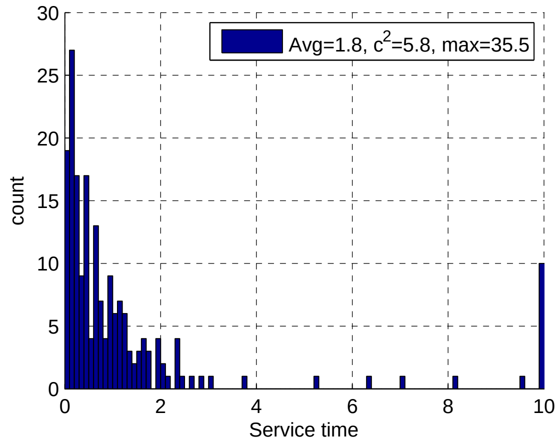

Figure 80: Outlier 1

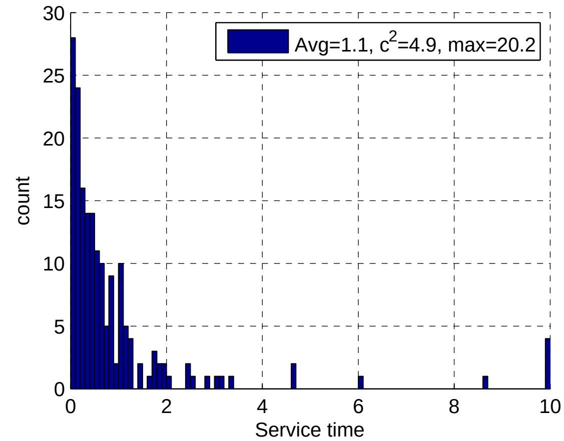

Figure 81: Outlier 2

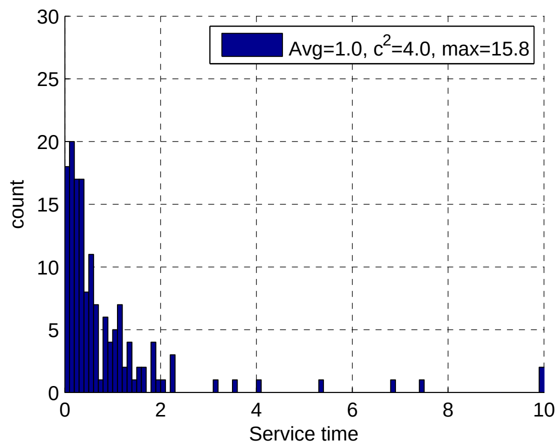

Figure 82: Outlier 3

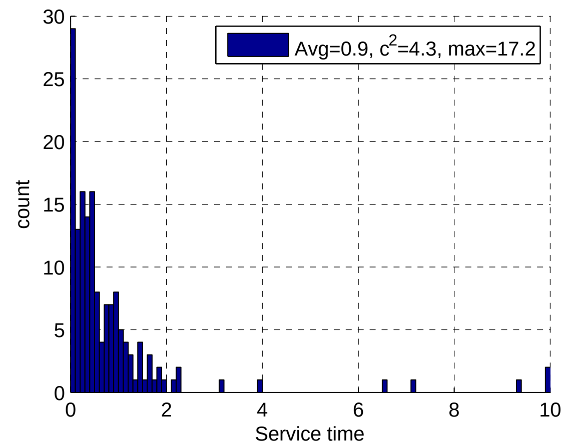

Figure 83: Outlier 4

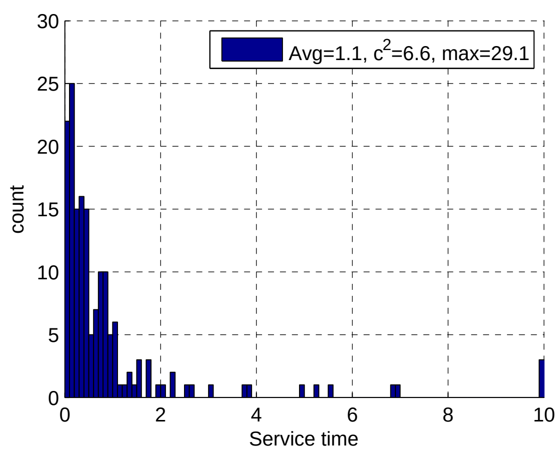

Figure 84: Outlier 5

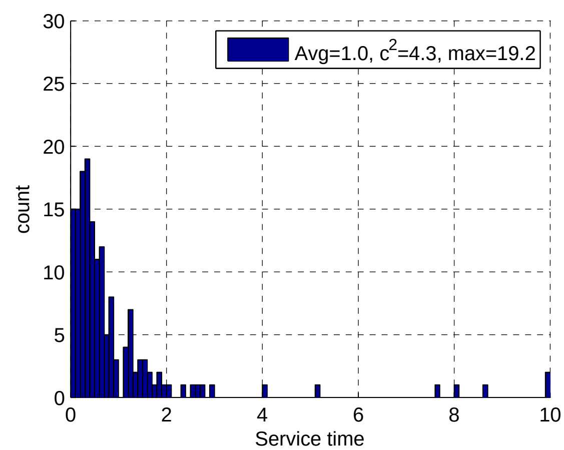

Figure 85: Outlier 6

Figures 80-85: Linear arrival rate and H_2 service: Histograms (bin size: 0.1) for service times in the interval $[0, 4]$ of sample paths with extreme values of $W_{L,\lambda,q,p}(t)$ ($W_{L,\lambda,q,p}(t) < 0$ or $W_{L,\lambda,q,p}(t) > 2$). Counts at 10 indicate # service times that are greater than 10.

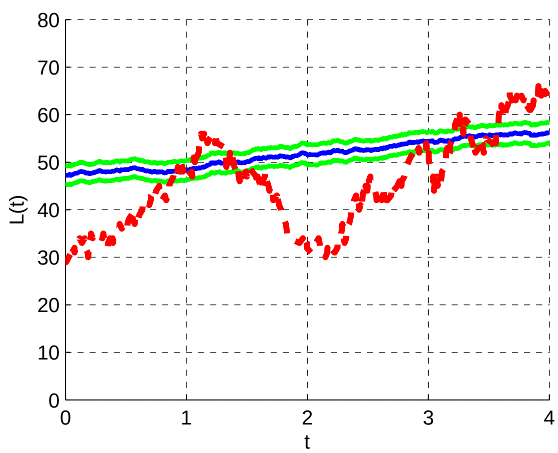

Figure 86: $L(t)$ (dashed line) as well as average $L(t)$ with associated 95% confidence interval based on 100 replications. $\beta = 0$.

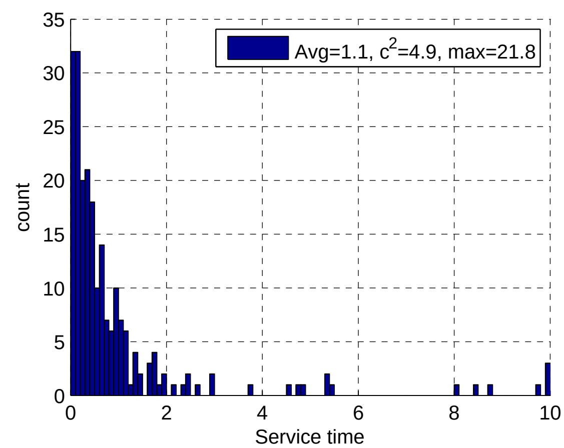

Figure 87: Histograms (bin size: 0.1) for service times in the interval $[0, 4]$. Counts at 10 indicate # service times that are greater than 10.

Figures 86-87: Quadratic arrival rate and H_2 service: 1 extreme value of $W_{L,\lambda,q,p}(t)$ on $[0, 4]$.

2.5 Estimation Results - 1000 replications for H_2 service

Anticipating that the poor performance of the estimators for the H_2 service time distribution in Section 2.3 is due to high variability in service times, we explore further by doing $1,000$ replications in this section. The results below support that the poor performance for the H_2 service time distribution is in part due to the small sample size. However, as explored in Section 2.4 and illustrated by the performance of $\bar{W}_{L,\lambda,q,p'}(t)$ in Table 19, 21 and 22, a small percentage of extreme values of $W_{L,\lambda,q,p}(t)$ are the main cause for the poor performance of quadratic estimators.

A	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
C	[0, 4]	0	1.034 ± 0.011	1.010 ± 0.009	1.011 ± 0.009	1.014 ± 0.013	1.012 ± 0.009	1.003 ± 0.009	1.003 ± 0.009	0.170 ± 0.082	1.016 ± 0.207	0.170 ± 0.082
		1	0.999 ± 0.010	0.967 ± 0.007	0.972 ± 0.007	0.980 ± 0.010	0.970 ± 0.007	0.962 ± 0.007	0.962 ± 0.007	0.240 ± 0.076	0.854 ± 0.051	0.854 ± 0.051
		2	0.994 ± 0.010	0.961 ± 0.006	0.966 ± 0.006	0.975 ± 0.009	0.963 ± 0.006	0.956 ± 0.006	0.956 ± 0.006	0.254 ± 0.075	0.863 ± 0.029	0.863 ± 0.029
	[0, 8]	0	1.057 ± 0.010	1.031 ± 0.009	1.042 ± 0.009	1.062 ± 0.012	1.032 ± 0.009	1.031 ± 0.009	1.032 ± 0.009	1.038 ± 0.009	1.028 ± 0.009	1.028 ± 0.009
		1	1.006 ± 0.008	0.980 ± 0.006	0.985 ± 0.006	0.996 ± 0.008	0.980 ± 0.006	0.979 ± 0.006	0.980 ± 0.006	0.983 ± 0.006	0.976 ± 0.006	0.976 ± 0.006
		2	0.996 ± 0.007	0.971 ± 0.005	0.975 ± 0.005	0.983 ± 0.007	0.972 ± 0.005	0.971 ± 0.005	0.972 ± 0.005	0.974 ± 0.006	0.967 ± 0.005	0.967 ± 0.005
		Avg	1.014	0.987	0.992	1.002	0.988	0.983	0.984	0.610	0.951	0.810
L	[0, 4]	0	1.052 ± 0.012	0.863 ± 0.008	0.915 ± 0.009	1.018 ± 0.013	1.177 ± 0.018	1.019 ± 0.011	1.019 ± 0.011	-0.65 ± 0.23	0.969 ± 0.219	0.969 ± 0.219
		1	1.008 ± 0.011	0.815 ± 0.006	0.868 ± 0.006	0.974 ± 0.010	1.070 ± 0.013	0.952 ± 0.007	0.952 ± 0.007	-0.47 ± 0.18	0.366 ± 0.888	-0.47 ± 0.18
		2	0.999 ± 0.011	0.806 ± 0.005	0.859 ± 0.006	0.965 ± 0.009	1.051 ± 0.011	0.941 ± 0.006	0.941 ± 0.006	-0.40 ± 0.17	0.852 ± 0.048	0.852 ± 0.048
	[0, 8]	0	1.058 ± 0.009	0.885 ± 0.007	0.940 ± 0.008	1.051 ± 0.011	1.135 ± 0.014	1.033 ± 0.010	1.033 ± 0.010	0.695 ± 0.106	1.029 ± 0.010	1.029 ± 0.010
		1	1.009 ± 0.008	0.835 ± 0.005	0.885 ± 0.005	0.986 ± 0.007	1.039 ± 0.008	0.966 ± 0.006	0.966 ± 0.006	0.942 ± 0.049	0.962 ± 0.007	0.962 ± 0.007
		2	0.999 ± 0.007	0.825 ± 0.004	0.875 ± 0.005	0.974 ± 0.006	1.022 ± 0.007	0.953 ± 0.005	0.953 ± 0.005	1.005 ± 0.022	0.949 ± 0.006	0.949 ± 0.006
		Avg	1.021	0.838	0.890	0.995	1.082	0.977	0.977	0.188	0.855	0.715
Q	[0, 4]	0	1.080 ± 0.011	0.931 ± 0.009	0.968 ± 0.010	1.042 ± 0.014	1.074 ± 0.013	1.029 ± 0.011	1.029 ± 0.011	-0.53 ± 0.11	1.035 ± 0.021	1.035 ± 0.021
		1	1.016 ± 0.010	0.868 ± 0.006	0.903 ± 0.007	0.972 ± 0.010	0.985 ± 0.009	0.953 ± 0.007	0.953 ± 0.007	-0.34 ± 0.10	0.953 ± 0.014	0.953 ± 0.014
		2	1.001 ± 0.009	0.856 ± 0.005	0.890 ± 0.005	0.957 ± 0.008	0.967 ± 0.007	0.938 ± 0.006	0.938 ± 0.006	-0.30 ± 0.10	0.935 ± 0.012	0.935 ± 0.012
	[0, 8]	0	1.068 ± 0.008	0.953 ± 0.007	0.980 ± 0.008	1.034 ± 0.010	1.092 ± 0.010	1.058 ± 0.009	1.058 ± 0.009	0.425 ± 0.098	1.080 ± 0.010	1.080 ± 0.010
		1	1.012 ± 0.007	0.900 ± 0.005	0.924 ± 0.005	0.974 ± 0.007	1.019 ± 0.007	0.992 ± 0.006	0.992 ± 0.006	0.825 ± 0.059	1.005 ± 0.007	1.005 ± 0.007
		2	0.999 ± 0.007	0.889 ± 0.004	0.913 ± 0.004	0.961 ± 0.006	1.004 ± 0.006	0.979 ± 0.005	0.979 ± 0.005	0.922 ± 0.042	0.989 ± 0.006	0.989 ± 0.006
		Avg	1.029	0.899	0.930	0.990	1.024	0.991	0.991	0.167	0.999	0.999

Table 18: H_2 service with 1000 replications: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 1000 replications.

		Constant			Linear			Quadratic		
Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$
[0, 4]	0	1.016 ± 0.207	25	0.949 ± 0.019	0.969 ± 0.219	47	0.963 ± 0.021	1.035 ± 0.021	14	1.039 ± 0.019
	1	0.854 ± 0.051	16	0.902 ± 0.016	0.366 ± 0.888	28	0.896 ± 0.016	0.953 ± 0.014	5	0.959 ± 0.013
	2	0.863 ± 0.029	14	0.894 ± 0.016	0.852 ± 0.048	26	0.884 ± 0.015	0.935 ± 0.012	5	0.940 ± 0.011
[0, 8]	0	1.028 ± 0.009	0	1.028 ± 0.009	1.029 ± 0.010	0	1.029 ± 0.010	1.079 ± 0.010	0	1.079 ± 0.010
	1	0.976 ± 0.006	0	0.976 ± 0.006	0.962 ± 0.007	0	0.962 ± 0.007	1.005 ± 0.007	0	1.005 ± 0.007
	2	0.967 ± 0.005	0	0.967 ± 0.005	0.949 ± 0.006	0	0.949 ± 0.006	0.989 ± 0.006	0	0.989 ± 0.006
	Avg	0.951	9.2	0.953	0.855	16.8	0.947	0.999	4.0	1.002

Table 19: H_2 service with 1000 replications: Waiting time estimates by $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ with associated 95% confidence intervals. $\bar{W}_{L,\lambda,q,p'}(t)$ is the new waiting time estimate after removing the outliers. The number of outliers removed in each case is given under $\#O$.

A	*Int*	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
C	[0, 4]	0	45.5 ± 0.4	$-4.09 \times 10^{-3} \pm 4.09 \times 10^{-3}$	$1.92 \times 10^{-2} \pm 1.51 \times 10^{-1}$
		1	43.5 ± 0.3	$-3.79 \times 10^{-3} \pm 3.91 \times 10^{-3}$	$1.09 \times 10^{-1} \pm 4.67 \times 10^{-2}$
		2	43.2 ± 0.3	$-3.70 \times 10^{-3} \pm 3.88 \times 10^{-3}$	$9.86 \times 10^{-2} \pm 3.22 \times 10^{-2}$
	[0, 8]	0	46.5 ± 0.4	$-4.89 \times 10^{-5} \pm 1.57 \times 10^{-3}$	$-1.42 \times 10^{-4} \pm 3.95 \times 10^{-3}$
		1	44.1 ± 0.3	$-1.59 \times 10^{-4} \pm 1.49 \times 10^{-3}$	$5.25 \times 10^{-4} \pm 3.54 \times 10^{-3}$
		2	43.7 ± 0.3	$-1.97 \times 10^{-4} \pm 1.47 \times 10^{-3}$	$5.98 \times 10^{-4} \pm 3.45 \times 10^{-3}$
		Avg	44.4	-2.00×10^{-3}	3.80×10^{-2}
L	[0, 4]	0	36.1 ± 0.4	$1.78 \times 10^{-1} \pm 3.12 \times 10^{-3}$	$5.18 \times 10^{-2} \pm 2.47 \times 10^{-1}$
		1	34.0 ± 0.3	$1.69 \times 10^{-1} \pm 2.80 \times 10^{-3}$	$6.99 \times 10^{-1} \pm 1.06 \times 10^{0}$
		2	33.6 ± 0.2	$1.67 \times 10^{-1} \pm 2.72 \times 10^{-3}$	$1.09 \times 10^{-1} \pm 5.86 \times 10^{-2}$
	[0, 8]	0	42.4 ± 0.4	$1.65 \times 10^{-1} \pm 1.48 \times 10^{-3}$	$4.26 \times 10^{-3} \pm 4.05 \times 10^{-3}$
		1	39.9 ± 0.3	$1.56 \times 10^{-1} \pm 1.09 \times 10^{-3}$	$3.81 \times 10^{-3} \pm 3.33 \times 10^{-3}$
		2	39.5 ± 0.2	$1.54 \times 10^{-1} \pm 9.66 \times 10^{-4}$	$3.95 \times 10^{-3} \pm 3.19 \times 10^{-3}$
		Avg	37.6	1.65×10^{-1}	1.45×10^{-1}
Q	[0, 4]	0	52.7 ± 0.6	$1.05 \times 10^{-1} \pm 3.24 \times 10^{-3}$	$-3.82 \times 10^{-2} \pm 2.18 \times 10^{-2}$
		1	49.1 ± 0.4	$9.76 \times 10^{-2} \pm 2.95 \times 10^{-3}$	$-3.17 \times 10^{-2} \pm 1.80 \times 10^{-2}$
		2	48.4 ± 0.3	$9.62 \times 10^{-2} \pm 2.87 \times 10^{-3}$	$-2.84 \times 10^{-2} \pm 1.73 \times 10^{-2}$
	[0, 8]	0	55.5 ± 0.5	$1.08 \times 10^{-1} \pm 1.21 \times 10^{-3}$	$-9.43 \times 10^{-2} \pm 3.14 \times 10^{-3}$
		1	52.3 ± 0.3	$1.02 \times 10^{-1} \pm 1.01 \times 10^{-3}$	$-8.24 \times 10^{-2} \pm 2.52 \times 10^{-3}$
		2	51.6 ± 0.3	$1.01 \times 10^{-1} \pm 9.21 \times 10^{-4}$	$-7.98 \times 10^{-2} \pm 2.34 \times 10^{-3}$
		Avg	51.6	1.02×10^{-1}	-5.91×10^{-2}

Table 20: H_2 service with 1000 replications: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 1000 replications.

A	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
C	$[0,4]$	0	2.4	2.3	2.0	2.1	3.1	3.1	86.4	1.7	86.4	8.4
		1	3.2	2.8	1.9	3.0	3.8	3.8	76.0	14.5	14.5	9.8
		2	3.3	2.8	1.9	3.1	3.8	3.8	74.0	13.1	13.1	10.0
	$[0,8]$	0	2.6	1.6	0.5	2.5	2.6	2.5	1.9	3.0	3.0	3.0
		1	2.6	2.0	1.0	2.5	2.6	2.5	2.2	3.0	3.0	3.0
		2	2.5	2.1	1.4	2.5	2.6	2.5	2.2	2.9	2.9	2.9
		Avg	2.8	2.3	1.4	2.6	3.1	3.0	40.4	6.4	20.5	6.2
L	$[0,4]$	0	18.9	13.7	3.4	12.5	3.3	3.3	169.9	8.3	8.3	8.9
		1	19.3	14.0	3.4	6.2	5.5	5.5	147.9	64.2	147.9	11.1
		2	19.3	14.0	3.3	5.2	5.8	5.8	139.3	14.6	14.6	11.5
	$[0,8]$	0	17.3	11.8	0.7	7.7	2.5	2.5	36.3	2.9	2.9	2.9
		1	17.4	12.4	2.3	3.0	4.3	4.3	6.7	4.7	4.7	4.7
		2	17.4	12.4	2.5	2.3	4.6	4.6	0.6	4.9	4.9	4.9
		Avg	18.3	13.0	2.6	6.2	4.3	4.3	83.5	16.6	30.5	7.3
Q	$[0,4]$	0	14.9	11.2	3.8	0.6	5.1	5.1	161.1	4.5	4.5	4.0
		1	14.8	11.4	4.4	3.1	6.3	6.3	135.7	6.4	6.4	5.7
		2	14.5	11.1	4.4	3.4	6.3	6.3	129.8	6.6	6.6	6.1
	$[0,8]$	0	11.4	8.7	3.4	2.5	0.9	0.9	64.3	1.2	1.2	1.2
		1	11.3	8.8	3.8	0.7	2.0	2.0	18.8	0.8	0.8	0.8
		2	11.1	8.6	3.8	0.5	2.0	2.0	7.8	1.0	1.0	1.0
		Avg	13.0	10.0	3.9	1.8	3.8	3.8	86.2	3.4	3.4	3.1

Table 21: H_2 service with 1000 replications: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

A	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
C	$[0,4]$	0	13.1 ± 0.6	11.7 ± 0.5	14.7 ± 0.7	13.2 ± 0.6	13.1 ± 0.6	89.1 ± 7.7	35.5 ± 17.1	23.5 ± 1.2
		1	13.3 ± 0.6	12.1 ± 0.6	13.4 ± 0.7	13.4 ± 0.6	13.4 ± 0.6	84.1 ± 7.6	27.2 ± 4.6	22.8 ± 1.2
		2	13.3 ± 0.6	12.1 ± 0.6	13.1 ± 0.6	13.5 ± 0.6	13.4 ± 0.6	82.9 ± 7.6	25.7 ± 2.6	22.7 ± 1.2
	$[0,8]$	0	8.6 ± 0.4	7.6 ± 0.4	8.8 ± 0.4	8.5 ± 0.4	8.4 ± 0.4	8.8 ± 0.4	8.7 ± 0.4	8.7 ± 0.4
		1	8.8 ± 0.4	8.0 ± 0.4	8.2 ± 0.4	8.6 ± 0.4	8.6 ± 0.4	8.7 ± 0.4	8.8 ± 0.4	8.8 ± 0.4
		2	8.8 ± 0.4	8.0 ± 0.4	8.0 ± 0.4	8.6 ± 0.4	8.6 ± 0.4	8.7 ± 0.4	8.7 ± 0.4	8.7 ± 0.4
		Avg	11.0	9.9	11.0	11.0	10.9	47.1	19.1	15.9
L	$[0,4]$	0	18.4 ± 0.7	14.5 ± 0.6	13.4 ± 0.6	21.0 ± 1.2	13.4 ± 0.6	170.6 ± 19.4	50.1 ± 22.2	24.6 ± 1.3
		1	19.1 ± 0.7	15.1 ± 0.6	12.7 ± 0.6	17.6 ± 1.0	13.4 ± 0.6	157.6 ± 16.0	72.6 ± 84.7	22.4 ± 1.2
		2	19.2 ± 0.7	15.2 ± 0.6	12.5 ± 0.6	17.1 ± 0.9	13.3 ± 0.6	151.3 ± 15.5	28.3 ± 4.4	21.9 ± 1.2
	$[0,8]$	0	16.0 ± 0.5	11.4 ± 0.4	7.8 ± 0.4	12.0 ± 0.6	8.5 ± 0.4	45.2 ± 8.5	8.9 ± 0.4	8.9 ± 0.4
		1	16.8 ± 0.5	12.3 ± 0.4	7.5 ± 0.3	9.6 ± 0.5	8.6 ± 0.4	18.5 ± 4.2	8.9 ± 0.4	8.9 ± 0.4
		2	16.8 ± 0.5	12.4 ± 0.4	7.5 ± 0.3	9.2 ± 0.4	8.6 ± 0.4	11.7 ± 2.0	8.9 ± 0.4	8.9 ± 0.4
		Avg	17.7	13.5	10.2	14.4	11.0	92.5	29.6	15.9
Q	$[0,4]$	0	15.0 ± 0.6	12.4 ± 0.5	12.6 ± 0.6	12.9 ± 0.6	11.9 ± 0.5	150.5 ± 10.2	20.5 ± 1.3	19.1 ± 1.0
		1	15.5 ± 0.6	12.8 ± 0.5	11.7 ± 0.5	12.4 ± 0.6	12.1 ± 0.5	138.6 ± 9.5	17.8 ± 1.0	17.3 ± 0.9
		2	15.4 ± 0.6	12.9 ± 0.5	11.5 ± 0.5	12.2 ± 0.6	12.1 ± 0.5	135.7 ± 9.5	17.3 ± 0.9	16.8 ± 0.8
	$[0,8]$	0	11.3 ± 0.4	9.1 ± 0.4	7.6 ± 0.3	8.7 ± 0.4	7.8 ± 0.4	67.1 ± 8.2	8.6 ± 0.4	8.6 ± 0.4
		1	11.6 ± 0.4	9.5 ± 0.4	7.2 ± 0.3	8.0 ± 0.4	7.7 ± 0.4	29.5 ± 5.1	7.9 ± 0.4	7.9 ± 0.4
		2	11.5 ± 0.4	9.5 ± 0.4	7.2 ± 0.3	7.8 ± 0.4	7.6 ± 0.4	20.2 ± 3.8	7.7 ± 0.4	7.7 ± 0.4
		Avg	13.4	11.0	9.6	10.4	9.9	90.2	13.3	12.9

Table 22: H_2 service with 1000 replications: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 1000 replications.

2.6 Longer Service Times

Formulas (4.6) and (4.9) show that the bias in $\bar{W}_{L,\lambda}(t)$ should be proportional to $E[W]$. Thus there should be more bias in $\bar{W}_{L,\lambda}(t)$ and we should achieve more bias reduction with longer service times. We illustrate that now by assuming that $E[S] = 4$ instead of 1. However, for these longer service times, the linear and quadratic approximations become less appropriate. Hence, we now use Theorem 2 as well as the other methods to do the estimation. We consider the previous case of the linear arrival rate function with exponential service. Since the system starts empty at time -12, the linear approximation is valid three mean service times in the past, and so should still be reasonable.

Table 23 provides different estimator values and Table 24 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (4.6) and (7.6)). Then, Tables 25 and 26 quantify the performance of our estimators by two performance measures. In all tables, we include the results for $E[S] = 1$ as well, to compare with the results for $E[S] = 4$.

$E[S]$	$Int.$	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
1	[0, 4]	0	1.038 ± 0.019	0.980 ± 0.020	1.047 ± 0.020	1.058 ± 0.023	1.058 ± 0.023	1.046 ± 0.022	1.046 ± 0.022	1.062 ± 0.021	1.046 ± 0.021	1.046 ± 0.021
		1	1.002 ± 0.016	0.939 ± 0.015	1.005 ± 0.015	1.011 ± 0.016	1.011 ± 0.016	1.000 ± 0.016	1.011 ± 0.016	1.014 ± 0.015	1.000 ± 0.015	1.014 ± 0.015
		2	0.996 ± 0.016	0.933 ± 0.014	1.000 ± 0.014	1.003 ± 0.015	1.003 ± 0.015	0.993 ± 0.015	1.003 ± 0.015	1.007 ± 0.013	0.993 ± 0.014	1.007 ± 0.013
	[0, 8]	0	1.051 ± 0.013	0.983 ± 0.015	1.050 ± 0.015	1.052 ± 0.017	1.052 ± 0.017	1.044 ± 0.016	1.044 ± 0.016	1.054 ± 0.016	1.045 ± 0.016	1.045 ± 0.016
		1	1.010 ± 0.010	0.944 ± 0.011	1.006 ± 0.011	1.008 ± 0.012	1.008 ± 0.012	1.001 ± 0.012	1.001 ± 0.012	1.009 ± 0.011	1.002 ± 0.011	1.009 ± 0.011
		2	1.003 ± 0.009	0.938 ± 0.010	0.998 ± 0.010	1.000 ± 0.011	1.000 ± 0.011	0.993 ± 0.011	0.993 ± 0.011	1.002 ± 0.010	0.994 ± 0.010	1.002 ± 0.010
	Avg		1.017	0.953	1.018	1.022	1.022	1.013	1.016	1.025	1.013	1.021
4	[0, 4]	0	3.954 ± 0.058	2.841 ± 0.041	3.617 ± 0.043	3.865 ± 0.065	4.000 ± 0.096	3.391 ± 0.047	3.391 ± 0.047	1.173 ± 7.752	3.048 ± 0.371	3.048 ± 0.371
		1	4.015 ± 0.063	2.877 ± 0.044	3.674 ± 0.047	3.938 ± 0.070	4.085 ± 0.102	3.441 ± 0.050	3.441 ± 0.050	1.207 ± 7.753	3.067 ± 0.374	3.067 ± 0.374
		2	4.034 ± 0.064	2.891 ± 0.046	3.696 ± 0.050	3.965 ± 0.072	4.118 ± 0.105	3.460 ± 0.052	3.460 ± 0.052	0.840 ± 7.782	3.085 ± 0.375	3.085 ± 0.375
	[0, 8]	0	3.948 ± 0.040	2.962 ± 0.025	3.702 ± 0.031	3.901 ± 0.036	3.937 ± 0.038	3.513 ± 0.030	3.513 ± 0.030	3.945 ± 0.038	3.490 ± 0.033	3.490 ± 0.033
		1	3.992 ± 0.041	2.998 ± 0.029	3.750 ± 0.035	3.968 ± 0.044	4.009 ± 0.047	3.562 ± 0.035	3.562 ± 0.035	3.838 ± 0.348	3.538 ± 0.038	3.538 ± 0.038
		2	4.003 ± 0.042	3.010 ± 0.030	3.764 ± 0.037	3.991 ± 0.047	4.034 ± 0.050	3.579 ± 0.038	3.579 ± 0.038	3.862 ± 0.350	3.555 ± 0.040	3.555 ± 0.040
	Avg		3.991	2.930	3.700	3.938	4.030	3.491	3.491	2.477	3.297	3.297

Table 23: $E[S] = 1$ and $E[S] = 4$: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

$E[S]$	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
1	$[0,4]$	0	40.8 ± 1.1	$6.72\times10^{-2} \pm 3.06\times10^{-3}$	$3.91\times10^{-3} \pm 4.79\times10^{-3}$
		1	39.1 ± 0.8	$6.46\times10^{-2} \pm 2.94\times10^{-3}$	$3.69\times10^{-3} \pm 4.32\times10^{-3}$
		2	38.8 ± 0.8	$6.42\times10^{-2} \pm 2.91\times10^{-3}$	$3.72\times10^{-3} \pm 4.26\times10^{-3}$
	$[0,8]$	0	47.2 ± 0.9	$6.17\times10^{-2} \pm 1.11\times10^{-3}$	$5.35\times10^{-4} \pm 7.53\times10^{-4}$
		1	45.3 ± 0.7	$5.93\times10^{-2} \pm 9.63\times10^{-4}$	$5.69\times10^{-4} \pm 6.81\times10^{-4}$
		2	45.0 ± 0.6	$5.89\times10^{-2} \pm 9.16\times10^{-4}$	$5.80\times10^{-4} \pm 6.69\times10^{-4}$
	Avg		42.7	6.27×10^{-2}	2.17×10^{-3}
4	$[0,4]$	0	118.0 ± 1.6	$1.95\times10^{-1} \pm 8.20\times10^{-3}$	$1.38\times10^{-1} \pm 1.42\times10^{-1}$
		1	119.5 ± 1.8	$1.97\times10^{-1} \pm 8.22\times10^{-3}$	$1.48\times10^{-1} \pm 1.43\times10^{-1}$
		2	120.1 ± 1.9	$1.98\times10^{-1} \pm 8.21\times10^{-3}$	$1.48\times10^{-1} \pm 1.43\times10^{-1}$
	$[0,8]$	0	142.0 ± 1.4	$1.86\times10^{-1} \pm 2.28\times10^{-3}$	$1.46\times10^{-2} \pm 1.06\times10^{-2}$
		1	143.8 ± 1.7	$1.88\times10^{-1} \pm 2.32\times10^{-3}$	$1.46\times10^{-2} \pm 1.09\times10^{-2}$
		2	144.4 ± 1.8	$1.89\times10^{-1} \pm 2.34\times10^{-3}$	$1.46\times10^{-2} \pm 1.10\times10^{-2}$
	Avg		131.3	1.92×10^{-1}	7.97×10^{-2}

Table 24: $E[S] = 1$ and $E[S] = 4$: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

$E[S]$	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
1	$[0,4]$	0	5.9	0.8	2.0	2.0	0.7	0.7	2.3	0.7	0.7
		1	6.3	0.3	0.9	0.8	0.2	0.8	1.2	0.2	1.2
		2	6.3	0.4	0.7	0.7	0.3	0.7	1.1	0.3	1.1
	$[0,8]$	0	6.8	0.1	0.1	0.1	0.7	0.7	0.3	0.6	0.6
		1	6.6	0.5	0.3	0.3	1.0	1.0	0.1	0.9	0.1
		2	6.5	0.4	0.3	0.3	1.0	1.0	0.1	0.8	0.1
	Avg		6.4	0.4	0.7	0.7	0.7	0.8	0.8	0.6	0.6
4	$[0,4]$	0	111.3	33.7	8.8	4.6	56.3	56.3	278.1	90.6	90.6
		1	113.8	34.1	7.7	7.0	57.5	57.5	280.9	94.8	94.8
		2	114.3	33.9	6.9	8.3	57.5	57.5	319.5	95.0	95.0
	$[0,8]$	0	98.6	24.6	4.7	1.1	43.5	43.5	0.3	45.8	45.8
		1	99.4	24.2	2.4	1.7	43.0	43.0	15.3	45.3	45.3
		2	99.3	23.9	1.1	3.1	42.4	42.4	14.1	44.8	44.8
	Avg		106.1	29.1	5.3	4.3	50.0	50.0	151.4	69.4	69.4

Table 25: $E[S] = 1$ and $E[S] = 4$: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

$E[S]$	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
1	[0, 4]	0	7.0 ± 1.1	4.3 ± 0.6	6.6 ± 0.9	6.6 ± 0.9	6.3 ± 0.8	6.2 ± 0.9	5.8 ± 0.8
		1	7.2 ± 1.1	4.4 ± 0.6	6.1 ± 0.9	6.1 ± 0.9	5.8 ± 0.8	5.7 ± 0.8	5.3 ± 0.7
		2	7.1 ± 1.1	4.4 ± 0.6	6.0 ± 0.8	6.0 ± 0.8	5.7 ± 0.7	5.6 ± 0.8	5.2 ± 0.7
	[0, 8]	0	6.6 ± 0.7	2.0 ± 0.3	3.4 ± 0.5	3.4 ± 0.6	3.3 ± 0.6	3.0 ± 0.5	3.1 ± 0.5
		1	6.6 ± 0.7	1.9 ± 0.3	3.1 ± 0.5	3.1 ± 0.5	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
		2	6.5 ± 0.6	1.9 ± 0.3	3.0 ± 0.4	3.0 ± 0.4	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
	Avg		6.8	3.2	4.7	4.7	4.5	4.3	4.1
4	[0, 4]	0	27.8 ± 1.5	9.9 ± 1.1	9.8 ± 1.4	11.9 ± 2.0	14.7 ± 1.5	267.1 ± 196.4	30.7 ± 8.6
		1	27.9 ± 1.6	10.0 ± 1.1	10.2 ± 1.5	12.3 ± 2.1	15.0 ± 1.5	265.2 ± 196.3	31.4 ± 8.5
		2	27.9 ± 1.6	9.9 ± 1.1	10.1 ± 1.5	12.3 ± 2.2	14.9 ± 1.5	274.2 ± 196.7	31.3 ± 8.5
	[0, 8]	0	24.8 ± 0.9	6.6 ± 0.7	4.9 ± 0.7	4.9 ± 0.8	11.0 ± 0.9	5.0 ± 0.7	11.6 ± 1.0
		1	24.7 ± 0.9	6.5 ± 0.7	4.9 ± 0.8	5.1 ± 0.8	10.9 ± 0.9	9.5 ± 8.6	11.4 ± 1.0
		2	24.7 ± 0.9	6.4 ± 0.7	4.9 ± 0.8	5.1 ± 0.9	10.7 ± 0.9	9.4 ± 8.4	11.3 ± 1.0
	Avg		26.3	8.2	7.5	8.6	12.9	138.4	21.3

Table 26: $E[S] = 1$ and $E[S] = 4$: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

2.7 A Decreasing Arrival Rate Function

In this section, we consider a special case of a decreasing arrival rate function and thus decreasing staffing. We use a minor modification of the previous linear arrival rate function, with time reversed. Specifically, we use $\lambda(t) = 48 - 3t$ over [0,4] and [0,8]. Table 27 gives the average of the estimated parameters for the linear decreasing arrival rate function over 100 replications.

	Constant	Linear		Quadratic		
Int.	$\bar{\lambda}(t)$	a	b	a	b	c
$[-4, 4]$	42.6 ± 0.6	48.4 ± 0.5	-2.877 ± 0.187	48.6 ± 0.8	-2.877 ± 0.187	-0.026 ± 0.100
$[-8, 8]$	36.1 ± 0.4	48.2 ± 0.4	-3.018 ± 0.068	48.4 ± 0.5	-3.018 ± 0.068	-0.009 ± 0.018

Table 27: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-4, 4]$ and $[-8, 8]$ to the arrival data for linear and decreasing arrival rate function; Average and halfwidths of 95% confidence intervals over 100 replications.

Int.		[0, 4]						[0, 8]					
GI	β	#*dec*	*Pr(Delay)*	$E[\#v]$	#*dep*	$\#v$	$\%v$	#*dec*	*Pr(Delay)*	$E[\#v]$	#*dep*	$\#v$	$\%v$
M	0	12	0.68 ± 0.06	8.18 ± 0.69	180.7 ± 2.7	7.97 ± 0.68	4.36 ± 0.36	24	0.67 ± 0.04	16.09 ± 1.06	314.3 ± 3.7	15.61 ± 1.02	4.93 ± 0.30
	1	13	0.25 ± 0.05	3.19 ± 0.63	182.0 ± 2.8	3.00 ± 0.64	1.59 ± 0.34	26	0.23 ± 0.04	5.89 ± 0.93	313.4 ± 3.6	5.25 ± 0.90	1.63 ± 0.27
	2	14	0.05 ± 0.02	0.69 ± 0.24	182.5 ± 2.9	0.64 ± 0.26	0.33 ± 0.14	28	0.04 ± 0.01	1.11 ± 0.34	313.3 ± 3.6	0.96 ± 0.32	0.29 ± 0.10
E_4	0	12	0.67 ± 0.06	8.04 ± 0.67	181.0 ± 2.2	7.95 ± 0.67	4.33 ± 0.34	24	0.66 ± 0.04	15.86 ± 1.06	314.2 ± 3.4	15.24 ± 1.08	4.78 ± 0.31
	1	12	0.20 ± 0.04	2.45 ± 0.50	182.2 ± 2.5	2.35 ± 0.52	1.23 ± 0.26	26	0.19 ± 0.03	4.89 ± 0.81	313.2 ± 3.5	4.33 ± 0.74	1.34 ± 0.22
	2	13	0.04 ± 0.02	0.56 ± 0.21	182.5 ± 2.6	0.53 ± 0.22	0.28 ± 0.11	27	0.03 ± 0.01	0.93 ± 0.30	313.1 ± 3.5	0.71 ± 0.26	0.22 ± 0.08
H_2	0	12	0.37 ± 0.06	4.43 ± 0.74	177.6 ± 2.5	4.14 ± 0.73	2.30 ± 0.40	24	0.41 ± 0.05	9.83 ± 1.30	307.8 ± 3.5	9.42 ± 1.30	3.04 ± 0.41
	1	13	0.07 ± 0.02	0.88 ± 0.31	179.2 ± 2.6	0.83 ± 0.31	0.46 ± 0.17	26	0.09 ± 0.03	2.29 ± 0.71	308.6 ± 3.4	2.23 ± 0.77	0.72 ± 0.25
	2	14	0.01 ± 0.00	0.08 ± 0.06	179.5 ± 2.6	0.07 ± 0.06	0.04 ± 0.03	28	0.01 ± 0.01	0.27 ± 0.19	308.8 ± 3.4	0.29 ± 0.25	0.09 ± 0.08

Table 28: Early service termination in the 9 different $M_t/GI/s_t$ models with linear decreasing arrival rate and staffing according to (8.1) with QoS parameter β.#*dec* indicates the number of staffing decreases, #*dep* indicates the number of departures and v indicates violations. Associated 95% confidence intervals based on 100 replications are also shown.

We use three service time distributions, M, H_2 and E_4, with the same specifications given in the beginning of Section 2.2. We compute $m(t)$ and $s(t)$ as before, with $\beta = 0$, 1, and 2 and start empty at -12. This gives us nine different models. If a server is scheduled to depart when all servers are busy, then in our simulations we let that server depart immediately and force the customer with the least remaining service time to complete service at that time. In fact, we assume that the server scheduled to leave would actually depart only after that minimum remaining service time has elapsed. At that time, the server completing service can take over the service of the departing server's customer, because service switching is allowed.

Int.		[0, 4]					[0, 8]				
GI	β	$E[W]$	$\%Delayed$	$\%Aban.$	$\%EarlyTer.$	$TETT$	$E[W]$	$\%Delayed$	$\%Aban.$	$\%EarlyTer.$	$TETT$
M	0	1.12 ± 0.02	67.7 ± 5.7	4.15 ± 0.78	4.85 ± 0.42	0.20 ± 0.02	1.11 ± 0.02	65.4 ± 4.4	4.97 ± 0.74	5.35 ± 0.34	0.45 ± 0.04
	1	1.04 ± 0.02	23.7 ± 4.7	0.61 ± 0.21	1.61 ± 0.32	0.06 ± 0.02	1.03 ± 0.01	21.0 ± 3.4	0.66 ± 0.19	1.68 ± 0.27	0.12 ± 0.02
	2	1.02 ± 0.02	4.6 ± 1.6	0.04 ± 0.03	0.35 ± 0.14	0.01 ± 0.01	1.02 ± 0.01	3.4 ± 1.1	0.04 ± 0.03	0.32 ± 0.10	0.02 ± 0.01
E_4	0	1.10 ± 0.02	66.6 ± 5.6	3.05 ± 0.63	4.76 ± 0.38	0.20 ± 0.02	1.08 ± 0.01	64.3 ± 4.4	3.84 ± 0.62	5.17 ± 0.32	0.43 ± 0.04
	1	1.02 ± 0.01	19.5 ± 4.1	0.37 ± 0.14	1.39 ± 0.30	0.05 ± 0.01	1.01 ± 0.01	17.1 ± 2.8	0.39 ± 0.11	1.42 ± 0.22	0.10 ± 0.02
	2	1.01 ± 0.01	4.0 ± 1.5	0.04 ± 0.03	0.28 ± 0.12	0.01 ± 0.00	1.00 ± 0.01	2.9 ± 0.9	0.04 ± 0.03	0.23 ± 0.08	0.01 ± 0.00
H_2	0	1.05 ± 0.04	36.3 ± 6.2	1.93 ± 0.66	3.44 ± 0.49	2.09 ± 0.76	1.04 ± 0.03	41.0 ± 5.5	3.62 ± 1.02	4.13 ± 0.50	4.38 ± 1.25
	1	1.01 ± 0.04	6.5 ± 2.3	0.13 ± 0.09	0.91 ± 0.26	1.70 ± 0.66	1.00 ± 0.03	8.8 ± 2.9	0.52 ± 0.35	1.25 ± 0.29	3.80 ± 1.18
	2	1.01 ± 0.04	0.5 ± 0.4	0.01 ± 0.01	0.37 ± 0.12	1.51 ± 0.62	0.99 ± 0.03	1.0 ± 0.8	0.04 ± 0.06	0.54 ± 0.14	3.37 ± 1.12

Table 29: Average performance of the 9 different $M_t/GI/s_t$ models with linear decreasing arrival rate and staffing according to (8.1) with QoS parameter β, averaged over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. $TETT$ indicates the total early termination time. Associated 95% confidence intervals based on 100 replications are also shown.

To study this effect, Table 28 shows the number of staffing decreases ($\#dec$), the number of departures ($\#dep$), the number of violation ($\#v$) and the percentage of departures that are violations ($\%v$) in each case. From Table 28, we see that we could also estimate the number of violations in advance, before doing any simulation, by $\#dec \times P(W > 0)$, where $\#dec$ can be computed from the offered load and thus the staffing function given β, while $P(W > 0)$ can be estimated using the Garnett function for the stationary $M/M/s + M$ Erlang-A model in equation (11) and Figure 6 of [2]. However, in Table 28, $P(W > 0)$ is estimated from the simulation output by the number of arrivals that have to wait before starting service divided by the total number of arrivals during the interval.

Table 29 shows other key performance estimates, including the total early termination time ($TETT$), which can be divided by the number of arrivals to estimate the addition to the mean waiting time. For M service, $TETT$ can also be estimated in advance, before doing any simulation, by multiplying the number of violations (estimated as described above) by the the expected minimum remaining service time among all servers, which can be estimated by mean service time divided by the average number of service time, i.e., the expected minimum remaining service time among all servers servers. From the simulation results, we show that, for M and E_4 service, the average waiting time is consistently increased by about 0.1% for $\beta = 0$ and much less for $\beta = 1$ and 2. For H_2 service, the average waiting time is consistently increased by about 1.0% or less, which is still negligible. The H_2 case is relatively more problematic, because the remaining waiting times tend to be much longer than $E[W] \approx E[S] = 1$, usually having mean close to the larger of the two exponential means, Nevertheless, this effect is still relatively small. Figures 88 - 99 provide more information on the model performance over time.

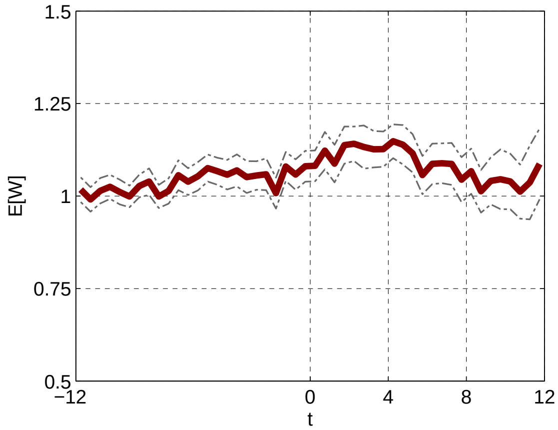

Figure 88: Average waiting time: QoS parameter $\beta = 0$.

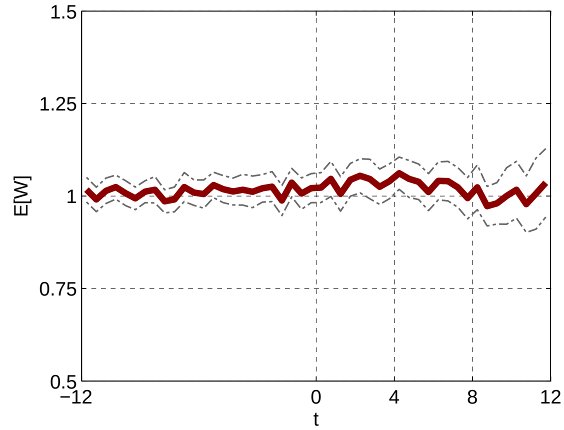

Figure 89: Average waiting time: QoS parameter $\beta = 1$.

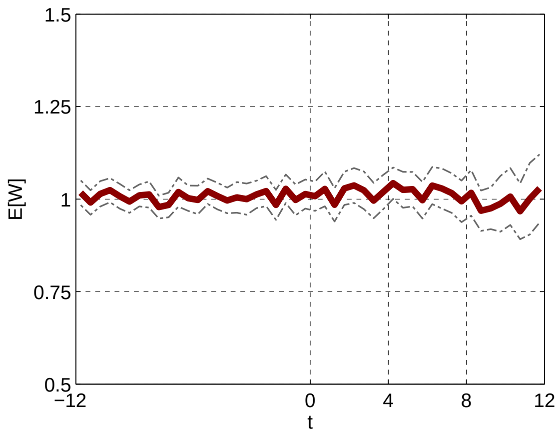

Figure 90: Average waiting time: QoS parameter $\beta = 2$.

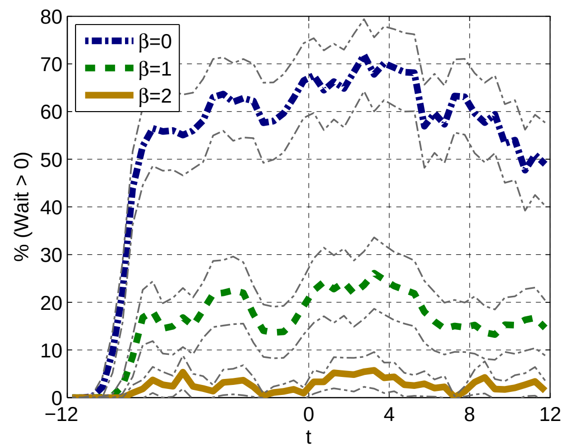

Figure 91: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

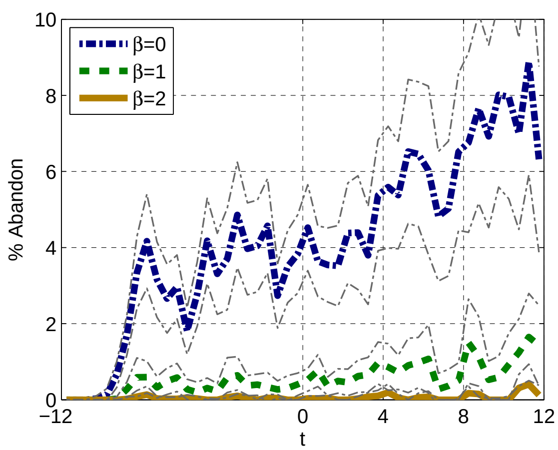

Figure 92: Average percent of abandonment: QoS parameter $\beta = 0$, 1 and 2.

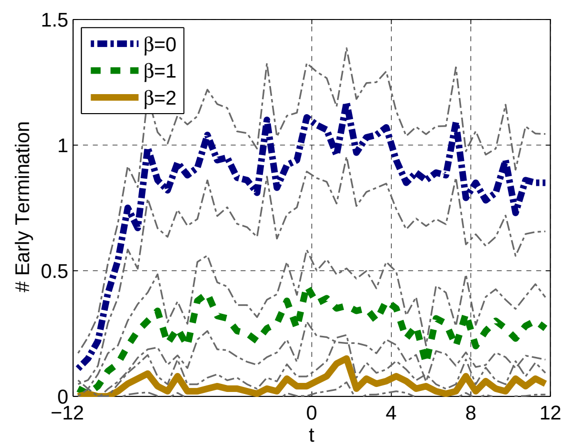

Figure 93: Number of early termination in each subinterval of length 0.5: QoS parameter $\beta = 0$, 1 and 2.

Figures 88-93: Linear (decreasing) arrival rate and M service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

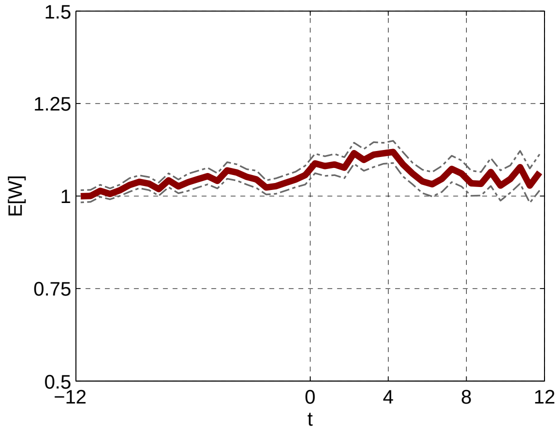

Figure 94: Average waiting time: QoS parameter $\beta = 0$.

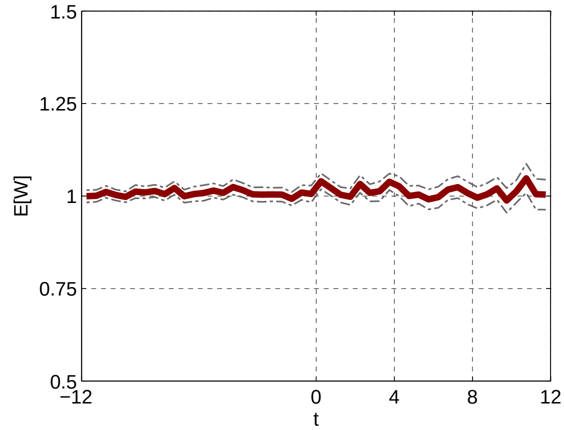

Figure 95: Average waiting time: QoS parameter $\beta = 1$.

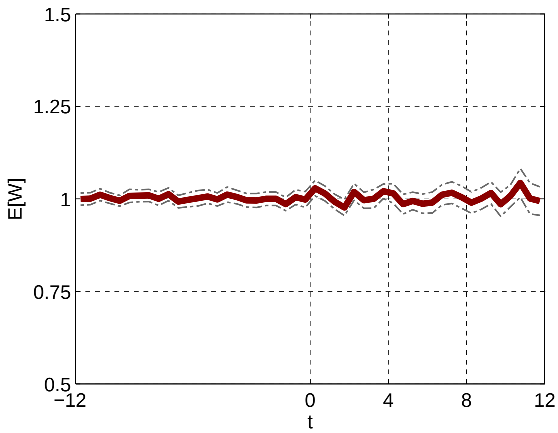

Figure 96: Average waiting time: QoS parameter $\beta = 2$.

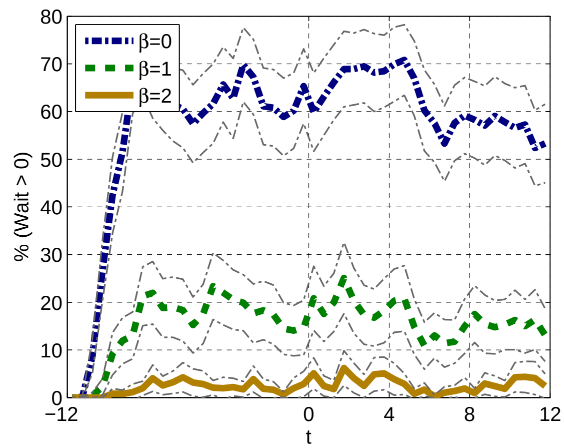

Figure 97: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

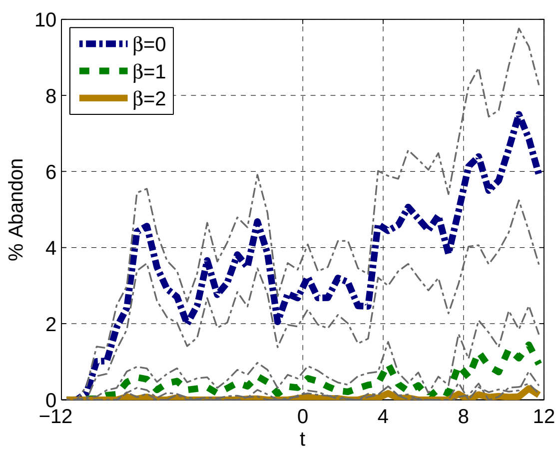

Figure 98: Average percent of arrivals abandoning: QoS parameter $\beta = 0$, 1 and 2.

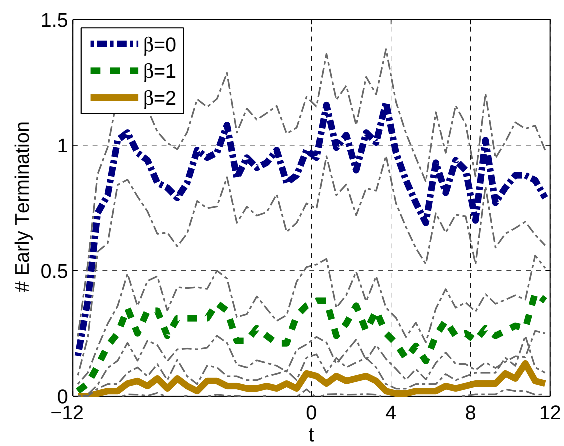

Figure 99: Number of early termination in each subinterval of length 0.5: QoS parameter $\beta = 0$, 1 and 2.

Figures 94-99: Linear (decreasing) arrival rate and E_4 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

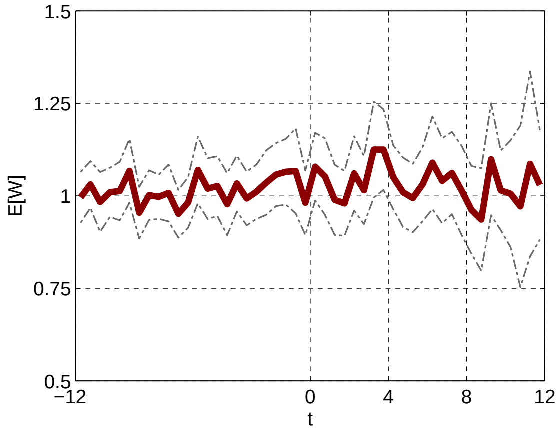

Figure 100: Average waiting time: QoS parameter $\beta = 0$.

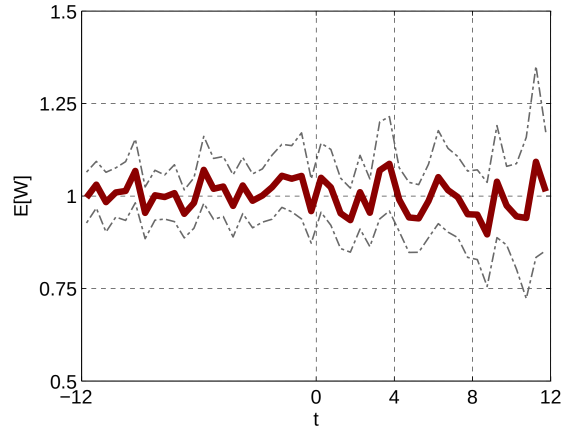

Figure 101: Average waiting time: QoS parameter $\beta = 1$.

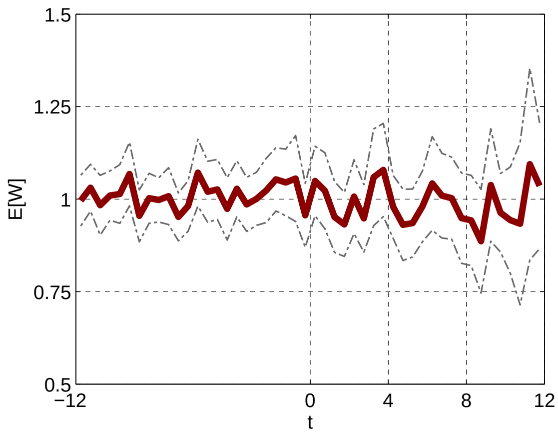

Figure 102: Average waiting time: QoS parameter $\beta = 2$.

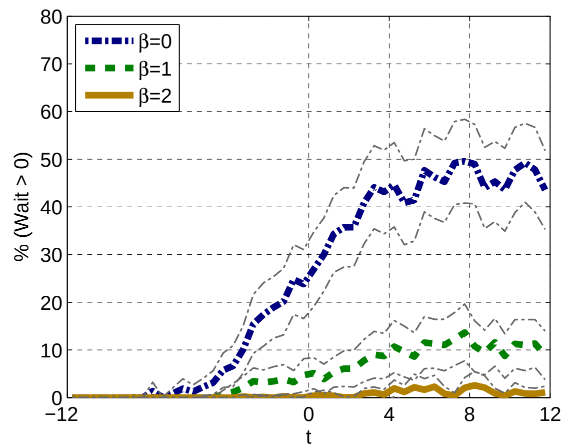

Figure 103: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

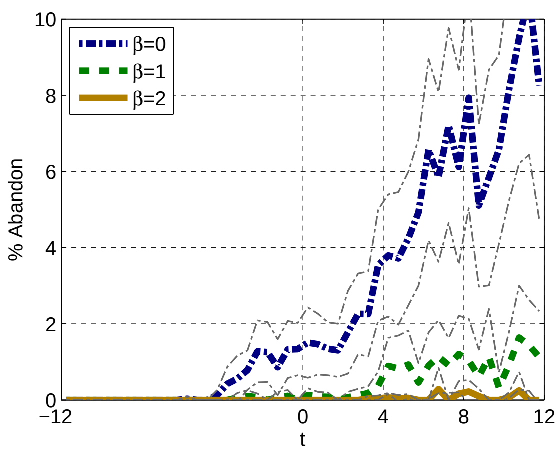

Figure 104: Average percent of abandonment: QoS parameter $\beta = 0$, 1 and 2.

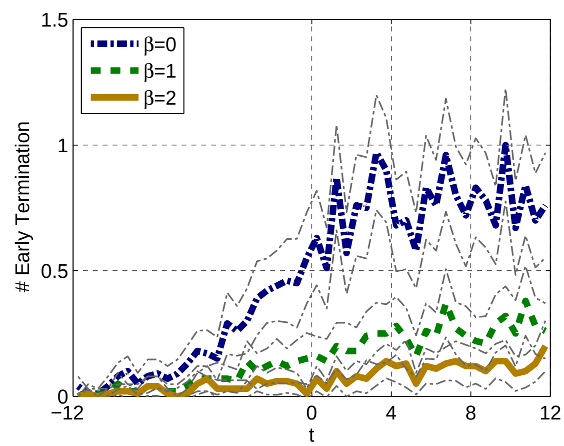

Figure 105: Number of early termination in each subinterval of length 0.5: QoS parameter $\beta = 0$, 1 and 2.

Figures 100-105: Linear (decreasing) arrival rate and H_2 service time distribution. Average performance over periods of length 0.5 with associated 95% confidence intervals based on 100 replications.

We now present estimation results. Table 30 provides different estimator values and Table 32 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (4.6) and (7.6)).

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.121 ± 0.024	1.183 ± 0.026	1.109 ± 0.027	1.109 ± 0.027	1.099 ± 0.022	1.085 ± 0.022	1.085 ± 0.022	1.098 ± 0.021	1.080 ± 0.020	1.080 ± 0.020
		1	1.037 ± 0.019	1.107 ± 0.019	1.030 ± 0.020	1.030 ± 0.020	1.032 ± 0.016	1.021 ± 0.016	1.021 ± 0.016	1.031 ± 0.015	1.017 ± 0.015	1.017 ± 0.015
		2	1.016 ± 0.016	1.089 ± 0.016	1.011 ± 0.017	1.011 ± 0.017	1.017 ± 0.014	1.006 ± 0.013	1.006 ± 0.013	1.016 ± 0.013	1.002 ± 0.013	1.016 ± 0.013
	[0, 8]	0	1.113 ± 0.019	1.203 ± 0.021	1.096 ± 0.018	1.096 ± 0.018	1.101 ± 0.018	1.081 ± 0.017	1.081 ± 0.017	1.100 ± 0.017	1.078 ± 0.016	1.078 ± 0.016
		1	1.035 ± 0.014	1.122 ± 0.015	1.026 ± 0.014	1.026 ± 0.014	1.033 ± 0.013	1.016 ± 0.013	1.016 ± 0.013	1.032 ± 0.013	1.014 ± 0.012	1.014 ± 0.012
		2	1.018 ± 0.012	1.104 ± 0.013	1.011 ± 0.012	1.011 ± 0.012	1.018 ± 0.011	1.002 ± 0.011	1.002 ± 0.011	1.017 ± 0.011	1.000 ± 0.011	1.000 ± 0.011
	Avg		1.057	1.135	1.047	1.047	1.050	1.035	1.035	1.049	1.032	1.034
E_4	[0, 4]	0	1.097 ± 0.017	1.129 ± 0.019	1.055 ± 0.018	1.083 ± 0.017	1.078 ± 0.017	1.074 ± 0.017	1.078 ± 0.017	1.076 ± 0.015	1.070 ± 0.015	1.070 ± 0.015
		1	1.020 ± 0.009	1.060 ± 0.011	0.984 ± 0.012	1.012 ± 0.010	1.015 ± 0.011	1.011 ± 0.010	1.011 ± 0.010	1.014 ± 0.010	1.009 ± 0.009	1.009 ± 0.009
		2	1.006 ± 0.008	1.048 ± 0.009	0.971 ± 0.010	1.000 ± 0.008	1.004 ± 0.009	1.000 ± 0.008	1.000 ± 0.008	1.003 ± 0.008	0.998 ± 0.008	0.998 ± 0.008
	[0, 8]	0	1.087 ± 0.013	1.143 ± 0.015	1.041 ± 0.013	1.079 ± 0.013	1.082 ± 0.015	1.074 ± 0.014	1.074 ± 0.014	1.081 ± 0.014	1.072 ± 0.013	1.072 ± 0.013
		1	1.017 ± 0.007	1.069 ± 0.009	0.978 ± 0.008	1.012 ± 0.007	1.016 ± 0.009	1.009 ± 0.008	1.009 ± 0.008	1.015 ± 0.008	1.008 ± 0.007	1.008 ± 0.007
		2	1.005 ± 0.006	1.057 ± 0.007	0.967 ± 0.007	1.001 ± 0.006	1.004 ± 0.007	0.998 ± 0.006	0.998 ± 0.006	1.003 ± 0.007	0.997 ± 0.006	0.997 ± 0.006
	Avg		1.039	1.084	0.999	1.031	1.033	1.028	1.028	1.032	1.026	1.026
H_2	[0, 4]	0	1.054 ± 0.040	1.174 ± 0.029	1.122 ± 0.031	1.017 ± 0.055	0.980 ± 0.024	0.885 ± 0.026	0.980 ± 0.024	0.396 ± 0.209	0.775 ± 0.095	0.775 ± 0.095
		1	1.010 ± 0.039	1.132 ± 0.020	1.072 ± 0.021	0.951 ± 0.038	0.950 ± 0.017	0.863 ± 0.021	0.950 ± 0.017	0.417 ± 0.198	0.768 ± 0.073	0.768 ± 0.073
		2	1.006 ± 0.038	1.128 ± 0.019	1.066 ± 0.020	0.943 ± 0.035	0.947 ± 0.016	0.860 ± 0.021	0.947 ± 0.016	0.415 ± 0.198	0.771 ± 0.067	0.771 ± 0.067
	[0, 8]	0	1.049 ± 0.033	1.247 ± 0.031	1.165 ± 0.031	1.000 ± 0.041	0.995 ± 0.021	0.846 ± 0.014	0.846 ± 0.014	0.997 ± 0.022	0.839 ± 0.014	0.839 ± 0.014
		1	1.001 ± 0.029	1.194 ± 0.023	1.112 ± 0.023	0.947 ± 0.030	0.961 ± 0.015	0.828 ± 0.011	0.828 ± 0.011	0.962 ± 0.015	0.822 ± 0.011	0.822 ± 0.011
		2	0.996 ± 0.028	1.188 ± 0.021	1.105 ± 0.020	0.940 ± 0.027	0.956 ± 0.014	0.826 ± 0.010	0.826 ± 0.010	0.957 ± 0.014	0.820 ± 0.010	0.820 ± 0.010
	Avg		1.019	1.177	1.107	0.966	0.965	0.851	0.896	0.691	0.799	0.799

Table 30: LINEAR decreasing arrival rate: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

		M			E_4			H_2		
Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$
$[0, 4]$	0	1.080 ± 0.020	0	1.080 ± 0.021	1.070 ± 0.015	0	1.070 ± 0.015	0.775 ± 0.095	4	0.849 ± 0.054
	1	1.017 ± 0.015	0	1.017 ± 0.015	1.009 ± 0.009	0	1.009 ± 0.010	0.768 ± 0.073	3	0.815 ± 0.047
	2	1.002 ± 0.013	0	1.002 ± 0.013	0.998 ± 0.008	0	0.998 ± 0.008	0.771 ± 0.067	3	0.813 ± 0.047
$[0, 8]$	0	1.078 ± 0.016	0	1.078 ± 0.017	1.072 ± 0.013	0	1.072 ± 0.013	0.839 ± 0.014	0	0.839 ± 0.014
	1	1.014 ± 0.012	0	1.014 ± 0.013	1.008 ± 0.007	0	1.008 ± 0.007	0.822 ± 0.011	0	0.822 ± 0.011
	2	1.000 ± 0.011	0	1.000 ± 0.011	0.997 ± 0.006	0	0.997 ± 0.006	0.820 ± 0.010	0	0.820 ± 0.010

Table 31: LINEAR decreasing arrival rate: Waiting time estimates by $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ with associated 95% confidence intervals. $\bar{W}_{L,\lambda,q,p'}(t)$ is the new waiting time estimate after removing the outliers ($W_{L,\lambda,q,p}(t) < 0$ or $W_{L,\lambda,q,p}(t) > 2$). The number of outliers removed in each case is given under $\#O$.

Tables 33 and 34 quantify the performance of our estimators by two performance measures. The first one is the absolute difference between the average of the estimate in interest over the 100 replications and the average of the direct estimate of mean waiting time over the 100 replications.

GI	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
M	$[0,4]$	0	50.5 ± 1.4	$-8.16\times10^{-2} \pm 6.38\times10^{-3}$	$-3.67\times10^{-4} \pm 5.61\times10^{-3}$
		1	47.1 ± 1.0	$-7.64\times10^{-2} \pm 5.96\times10^{-3}$	$4.50\times10^{-5} \pm 4.93\times10^{-3}$
		2	46.4 ± 0.8	$-7.53\times10^{-2} \pm 5.90\times10^{-3}$	$2.90\times10^{-4} \pm 4.78\times10^{-3}$
	$[0,8]$	0	43.5 ± 0.9	$-1.01\times10^{-1} \pm 3.19\times10^{-3}$	$-5.48\times10^{-4} \pm 1.21\times10^{-3}$
		1	40.5 ± 0.7	$-9.42\times10^{-2} \pm 2.88\times10^{-3}$	$-3.88\times10^{-4} \pm 1.07\times10^{-3}$
		2	39.9 ± 0.6	$-9.28\times10^{-2} \pm 2.86\times10^{-3}$	$-3.19\times10^{-4} \pm 1.04\times10^{-3}$
	Avg		44.6	-8.69×10^{-2}	-2.15×10^{-4}
E_4	$[0,4]$	0	48.2 ± 1.2	$-4.86\times10^{-2} \pm 3.74\times10^{-3}$	$-5.25\times10^{-4} \pm 1.74\times10^{-3}$
		1	45.2 ± 0.8	$-4.56\times10^{-2} \pm 3.46\times10^{-3}$	$-1.86\times10^{-4} \pm 1.52\times10^{-3}$
		2	44.6 ± 0.7	$-4.51\times10^{-2} \pm 3.44\times10^{-3}$	$-1.10\times10^{-4} \pm 1.48\times10^{-3}$
	$[0,8]$	0	41.3 ± 0.9	$-5.99\times10^{-2} \pm 1.71\times10^{-3}$	$-2.19\times10^{-4} \pm 3.66\times10^{-4}$
		1	38.6 ± 0.6	$-5.61\times10^{-2} \pm 1.62\times10^{-3}$	$-1.48\times10^{-4} \pm 3.20\times10^{-4}$
		2	38.2 ± 0.5	$-5.55\times10^{-2} \pm 1.63\times10^{-3}$	$-1.29\times10^{-4} \pm 3.12\times10^{-4}$
	Avg		42.7	-5.18×10^{-2}	-2.19×10^{-4}
H_2	$[0,4]$	0	50.0 ± 1.4	$-2.42\times10^{-1} \pm 1.91\times10^{-2}$	$8.00\times10^{-2} \pm 9.85\times10^{-2}$
		1	48.1 ± 0.9	$-2.35\times10^{-1} \pm 1.88\times10^{-2}$	$7.25\times10^{-2} \pm 8.52\times10^{-2}$
		2	48.0 ± 0.9	$-2.34\times10^{-1} \pm 1.87\times10^{-2}$	$6.91\times10^{-2} \pm 8.14\times10^{-2}$
	$[0,8]$	0	45.0 ± 1.2	$-3.15\times10^{-1} \pm 1.31\times10^{-2}$	$-1.35\times10^{-4} \pm 1.42\times10^{-2}$
		1	43.1 ± 0.9	$-3.02\times10^{-1} \pm 1.17\times10^{-2}$	$3.16\times10^{-4} \pm 1.33\times10^{-2}$
		2	42.8 ± 0.8	$-3.01\times10^{-1} \pm 1.15\times10^{-2}$	$2.28\times10^{-4} \pm 1.32\times10^{-2}$
	Avg		46.2	-2.72×10^{-1}	3.70×10^{-2}

Table 32: LINEAR decreasing arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

The second measure is the absolute relative error of the estimate in each sample path, averaged over 100 replications.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
M	$[0,4]$	0	6.2	1.2	1.2	2.2	3.6	3.6	2.3	4.1	4.1	4.1
		1	7.0	0.7	0.7	0.4	1.6	1.6	0.5	2.0	2.0	2.0
		2	7.3	0.6	0.6	0.1	1.0	1.0	0.0	1.4	0.0	1.4
	$[0,8]$	0	9.1	1.7	1.7	1.1	3.2	3.2	1.3	3.4	3.4	3.4
		1	8.7	0.9	0.9	0.3	1.9	1.9	0.4	2.1	2.1	2.1
		2	8.6	0.8	0.8	0.1	1.7	1.7	0.2	1.9	1.9	1.9
	Avg		7.8	1.0	1.0	0.7	2.2	2.2	0.8	2.5	2.3	2.5
E_4	$[0,4]$	0	3.2	4.2	1.4	1.9	2.3	1.9	2.1	2.7	2.7	2.7
		1	4.0	3.6	0.8	0.5	0.9	0.9	0.6	1.1	1.1	1.1
		2	4.1	3.5	0.7	0.3	0.6	0.6	0.3	0.8	0.8	0.8
	$[0,8]$	0	5.6	4.6	0.8	0.5	1.2	1.2	0.6	1.4	1.4	1.4
		1	5.3	3.8	0.4	0.1	0.7	0.7	0.2	0.9	0.9	0.9
		2	5.2	3.7	0.4	0.1	0.7	0.7	0.1	0.8	0.8	0.8
	Avg		4.6	3.9	0.7	0.5	1.1	1.0	0.7	1.3	1.3	1.3
H_2	$[0,4]$	0	12.0	6.8	3.7	7.4	16.9	7.4	65.8	27.9	27.9	20.5
		1	12.2	6.2	5.9	6.1	14.8	6.1	59.3	24.2	24.2	19.5
		2	12.2	6.1	6.3	5.9	14.5	5.9	59.1	23.4	23.4	19.3
	$[0,8]$	0	19.7	11.5	4.9	5.4	20.4	20.4	5.2	21.0	21.0	21.0
		1	19.3	11.1	5.3	4.0	17.3	17.3	3.9	17.9	17.9	17.9
		2	19.2	10.9	5.6	3.9	17.0	17.0	3.8	17.6	17.6	17.6
	Avg		15.8	8.8	5.3	5.4	16.8	12.3	32.8	22.0	22.0	19.3

Table 33: LINEAR decreasing arrival rate: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
M	$[0,4]$	0	7.2 ± 1.1	4.0 ± 0.5	4.0 ± 0.5	5.6 ± 0.7	5.8 ± 0.8	5.2 ± 0.7	5.8 ± 0.8	5.8 ± 0.8
		1	8.1 ± 1.2	3.9 ± 0.5	3.9 ± 0.5	5.5 ± 0.7	5.6 ± 0.8	5.2 ± 0.8	5.4 ± 0.8	5.4 ± 0.8
		2	8.4 ± 1.2	4.0 ± 0.5	4.0 ± 0.5	5.6 ± 0.8	5.6 ± 0.8	5.2 ± 0.8	5.3 ± 0.8	5.3 ± 0.8
	$[0,8]$	0	8.3 ± 0.8	2.4 ± 0.3	2.4 ± 0.3	3.1 ± 0.5	3.7 ± 0.6	3.3 ± 0.4	3.9 ± 0.5	3.9 ± 0.5
		1	8.6 ± 0.8	2.3 ± 0.3	2.3 ± 0.3	3.0 ± 0.5	3.2 ± 0.5	3.1 ± 0.4	3.3 ± 0.5	3.3 ± 0.5
		2	8.6 ± 0.8	2.2 ± 0.3	2.2 ± 0.3	3.1 ± 0.5	3.3 ± 0.5	3.2 ± 0.4	3.3 ± 0.5	3.3 ± 0.5
	Avg		8.2	3.1	3.1	4.3	4.5	4.2	4.5	4.5
E_4	$[0,4]$	0	5.2 ± 0.8	4.1 ± 0.5	2.4 ± 0.3	3.8 ± 0.5	4.1 ± 0.6	3.5 ± 0.5	3.9 ± 0.6	3.9 ± 0.6
		1	5.0 ± 0.8	4.0 ± 0.5	2.3 ± 0.3	3.6 ± 0.5	3.6 ± 0.5	3.1 ± 0.4	3.1 ± 0.5	3.1 ± 0.5
		2	5.1 ± 0.8	4.1 ± 0.5	2.3 ± 0.3	3.6 ± 0.5	3.5 ± 0.5	3.2 ± 0.4	3.1 ± 0.5	3.1 ± 0.5
	$[0,8]$	0	5.3 ± 0.6	4.2 ± 0.3	1.4 ± 0.2	2.2 ± 0.3	2.5 ± 0.4	2.2 ± 0.3	2.4 ± 0.3	2.4 ± 0.3
		1	5.2 ± 0.6	3.9 ± 0.4	1.4 ± 0.2	2.2 ± 0.3	2.3 ± 0.3	2.3 ± 0.3	2.0 ± 0.3	2.0 ± 0.3
		2	5.2 ± 0.6	3.9 ± 0.4	1.3 ± 0.2	2.3 ± 0.3	2.3 ± 0.3	2.3 ± 0.3	2.0 ± 0.3	2.0 ± 0.3
	Avg		5.2	4.0	1.9	3.0	3.1	2.8	2.8	2.8
H_2	$[0,4]$	0	19.3 ± 3.4	14.6 ± 2.6	16.0 ± 2.6	14.2 ± 1.9	17.8 ± 2.2	69.0 ± 20.0	31.5 ± 7.2	25.6 ± 3.8
		1	20.9 ± 3.8	16.0 ± 2.9	13.8 ± 2.4	15.2 ± 2.2	18.1 ± 2.3	68.7 ± 20.2	29.9 ± 6.0	26.1 ± 4.1
		2	21.0 ± 3.8	16.2 ± 2.9	14.1 ± 2.4	15.2 ± 2.2	18.1 ± 2.3	68.8 ± 20.3	29.6 ± 5.7	26.1 ± 4.0
	$[0,8]$	0	21.0 ± 2.6	13.8 ± 1.8	10.8 ± 1.5	9.8 ± 1.3	18.3 ± 2.0	9.9 ± 1.3	18.8 ± 2.1	18.8 ± 2.1
		1	21.8 ± 2.8	14.4 ± 2.0	10.1 ± 1.5	10.1 ± 1.4	16.9 ± 1.9	10.2 ± 1.4	17.3 ± 2.0	17.3 ± 2.0
		2	21.9 ± 2.8	14.5 ± 2.0	10.1 ± 1.5	10.1 ± 1.4	16.8 ± 1.9	10.2 ± 1.4	17.2 ± 2.0	17.2 ± 2.0
	Avg		21.0	14.9	12.5	12.4	17.6	39.5	24.0	21.8

Table 34: LINEAR decreasing arrival rate: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

2.8 Sinusoidal Arrival Rate Function

In Section 8.6 of the main paper, we consider a sinusoidal arrival rate function, in order to illustrate how the estimation procedure should apply for a realistic arrival rate function arising in applications, which is not exactly linear or quadratic. In this section, we provide additional information about the simulation experiments and results summarized in Section 8.6.

The arrival rate function is now $\lambda(t) = 40+25\sin(t/2)$ over the intervals $[0, 4]$ and $[0, 8]$, starting empty at time -36. Following previous experiments, we let $E[S] = 1$ and consider two service time distributions: exponential and hyperexponential. Assuming that the system starts empty in the infinite past, as in (10) of the main paper, an exact expression for the offered load for exponential service time is $m(t) = 40 + 20(\sin(t/2) - (1/2)\cos(t/2))$ by (15) of [1]. If we let the service time by hyperexponential H_2 with $c_W^2 = 5$ and balanced means, an exact expression for the offered load is $m(t) = 40 + 25(0.5242\sin(t/2) - 0.2897\cos(t/2))$ by (29) of [1] after correcting an error in (29); see the short appendix in the main paper.

Based on this explicit offered load formula, we consider three levels of staffing, according to (8.1) with QoS parameter $\beta = 0$, 1 and 2 as before. The arrival rate, offered load and staffing with $\beta = 1$ are shown in Figures 106 and 107 for the exponential and H_2 service time distribution, respectively.

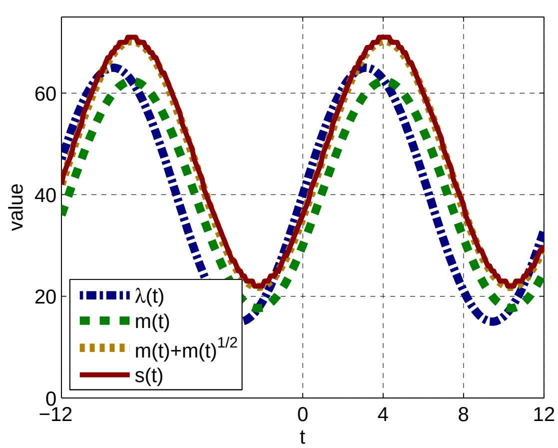

Figure 106: The sinusoidal arrival rate function, offered load function and staffing for exponential service time distribution according to (8.1) with $\beta = 1$ based on starting empty in the distant past. These will be applied over the intervals $[0, 4]$ and $[0, 8]$ to the system starting empty at time -36.

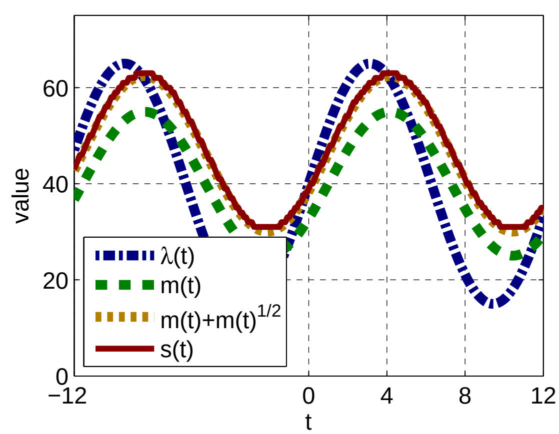

Figure 107: The sinusoidal arrival rate function, offered load function and staffing for H_2 service time distribution according to (8.1) with $\beta = 1$ based on starting empty in the distant past. These will be applied over the intervals $[0, 4]$ and $[0, 8]$ to the system starting empty at time -36.

Figures 106 and 107 show that a quadratic approximation to the arrival rate function should be appropriate over the target intervals $[0,4]$ and $[0,8]$. The arrival rate can be regarded as approximately nondecreasing over $[0,4]$, but it is decreasing over a significant portion of $[0,8]$, so that we also study the effect of server release when all servers are busy.

The experiment involves fitting a quadratic function to simulation data for the arrival process. For the target intervals $[0,4]$ and $[0,8]$, we base the estimation on data from the intervals $[-2,4]$ and $[-2,8]$, respectively, which supports a reasonable quadratic approximation. We simulate the $M_t/M/s_t+M$ and $M_t/H_2/s_t+M$ systems over the interval $[-36,12]$, starting empty at time -36.

Figure 35 illustrates the average of the estimated parameters for the sinusoidal arrival rate function over 100 replications and Figures 108 and 109 show plots of the approximations. We approximate the arrival rate by constant function, we consider intervals $[0,4]$ and $[0,8]$ instead of $[-2,4]$ and $[-2,8]$. The halfwidths of 95% confidence intervals for all estimates are also reported.

	Constant	Linear		Quadratic		
Int.	$\bar{\lambda}(t)$	a	b	a	b	c
$[-2,4]$	57.5 ± 0.7	39.0 ± 0.5	8.671 ± 0.312	41.7 ± 0.8	11.377 ± 0.433	-1.353 ± 0.195
$[-2,8]$	50.2 ± 0.5	44.4 ± 0.5	0.471 ± 0.116	43.1 ± 0.5	12.121 ± 0.326	-1.942 ± 0.051

Table 35: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,4]$ and $[-2,8]$ to the arrival data for sinusoidal arrival process; estimates and halfwidths of 95% confidence intervals over 100 replications.

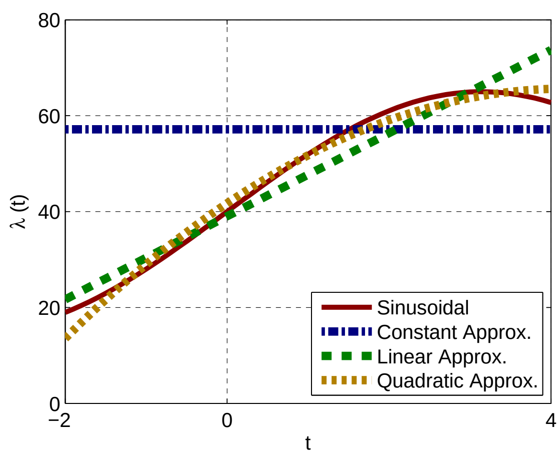

Figure 108: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,4]$ to the arrival data for sinusoidal arrival process.

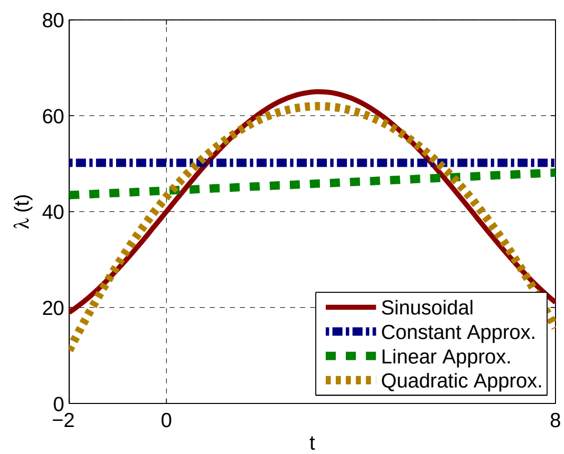

Figure 109: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,8]$ to the arrival data for sinusoidal arrival process.

Following previous examples, we now examine average performance of $M_t/M/s_t + M$ and $M_t/H_2/s_t + M$ systems used in our sinusoidal example. Table 37 illustrates $E[W]$, percent delayed, and percent abandoned. In each replication, we average the performance measures over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. The reported numbers are averages over the 100 replications and the halfwidths of 95% confidence intervals.

Int.		[0, 4]						[0, 8]					
GI	β	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$	$\%v$	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$	$\%v$
M	0	0	0.46 ± 0.05	0.00 ± 0.00	197.7 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	31	0.48 ± 0.04	14.84 ± 1.19	400.7 ± 3.7	0.00 ± 0.00	0.00 ± 0.00
	1	0	0.16 ± 0.03	0.00 ± 0.00	197.9 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	33	0.15 ± 0.03	4.93 ± 0.89	400.0 ± 3.6	0.00 ± 0.00	0.00 ± 0.00
	2	0	0.02 ± 0.01	0.00 ± 0.00	197.8 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	36	0.02 ± 0.01	0.79 ± 0.33	399.9 ± 3.6	0.00 ± 0.00	0.00 ± 0.00
H_2	0	0	0.48 ± 0.06	0.00 ± 0.00	207.0 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	8	0.49 ± 0.05	3.89 ± 0.44	400.3 ± 3.9	0.00 ± 0.00	0.00 ± 0.00
	1	0	0.16 ± 0.04	0.00 ± 0.00	207.7 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	8	0.17 ± 0.03	1.32 ± 0.28	399.3 ± 3.7	0.00 ± 0.00	0.00 ± 0.00
	2	0	0.02 ± 0.01	0.00 ± 0.00	207.8 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	9	0.02 ± 0.01	0.19 ± 0.08	399.1 ± 3.7	0.00 ± 0.00	0.00 ± 0.00

Table 36: Early service termination in the 9 different $M_t/GI/s_t$ models with sinusoidal arrival rate and staffing according to (8.1) with QoS parameter β.$\#dec$ indicates the number of staffing decreases, $\#dep$ indicates the number of departures and v indicates violations. Associated 95% confidence intervals based on 100 replications are also shown.

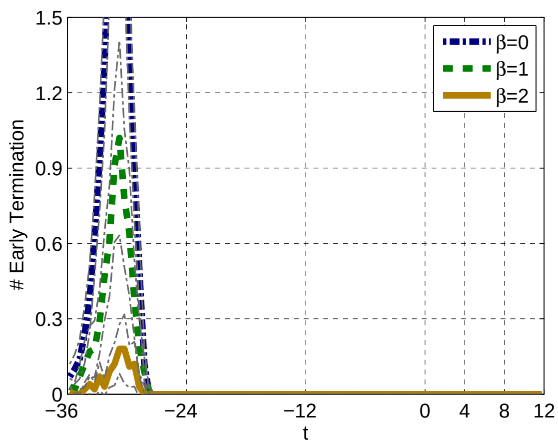

Figure 110: Number of early termination in each subinterval of length 0.5. M service with sinusoidal arrival rate.

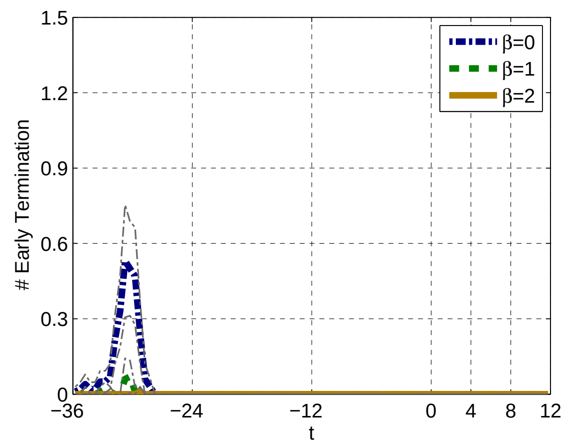

Figure 111: Number of early termination in each subinterval of length 0.5. H_2 service with sinusoidal arrival rate.

Figures 112 - 123 provide more information on the model performance over time. Performance of models with exponential service time distribution is shown in Figures 112 - 117. Model performance with hyperexponential service time distribution is shown in Figures 118 - 123. These plots suggest that the performance is stabilized approximately by time $t = 0$.

Performance Int.		[0, 4]			[0, 8]		
GI	β	$E[W]$	$\%Delayed$	$\%Aban.$	$E[W]$	$\%Delayed$	$\%Aban.$
M	0	1.03 ± 0.01	44.4 ± 4.5	3.66 ± 0.58	0.00 ± 0.00	0.0 ± 0.0	1.02 ± 0.01
	1	1.01 ± 0.01	14.5 ± 3.0	0.69 ± 0.23	0.00 ± 0.00	0.0 ± 0.0	1.00 ± 0.01
	2	1.01 ± 0.01	2.0 ± 1.0	0.05 ± 0.04	0.00 ± 0.00	0.0 ± 0.0	1.00 ± 0.01
H_2	0	1.04 ± 0.03	46.0 ± 5.9	4.26 ± 0.85	0.00 ± 0.00	0.0 ± 0.0	1.02 ± 0.02
	1	1.00 ± 0.03	14.9 ± 3.6	0.79 ± 0.27	0.00 ± 0.00	0.0 ± 0.0	0.99 ± 0.02
	2	1.00 ± 0.03	1.9 ± 0.9	0.04 ± 0.03	0.00 ± 0.00	0.0 ± 0.0	0.99 ± 0.02

Table 37: Average performance of the 6 different $M_t/GI/s_t$ models with sinusoidal arrival rate and staffing according to (8.1) with QoS parameter β, averaged over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. $TETT$ indicates the total early termination time. Associated 95% confidence intervals based on 100 replications are also shown.

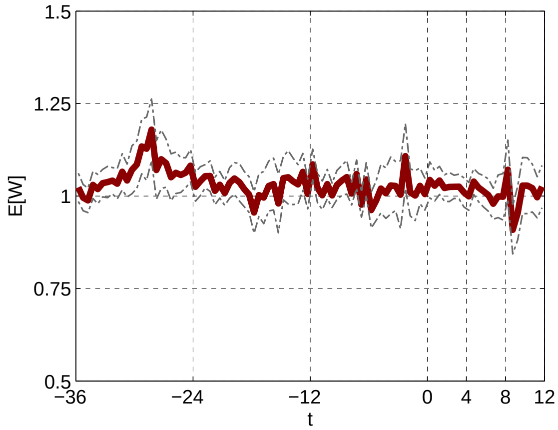

Figure 112: Average waiting time: QoS parameter $\beta = 0$.

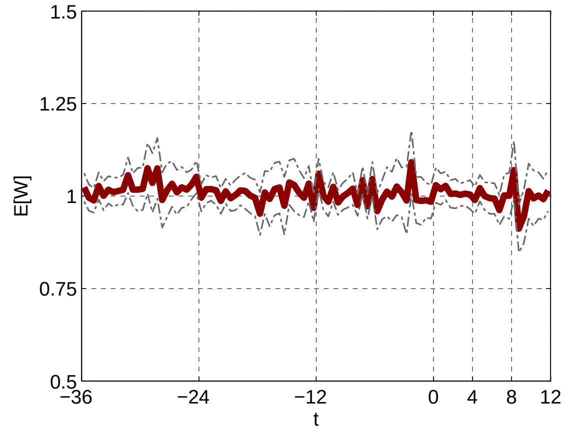

Figure 113: Average waiting time: QoS parameter $\beta = 1$.

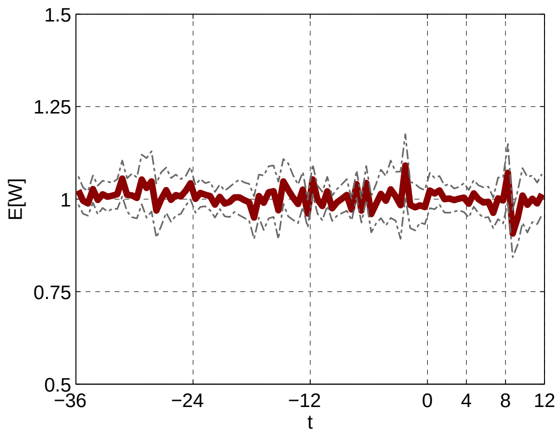

Figure 114: Average waiting time: QoS parameter $\beta = 2$.

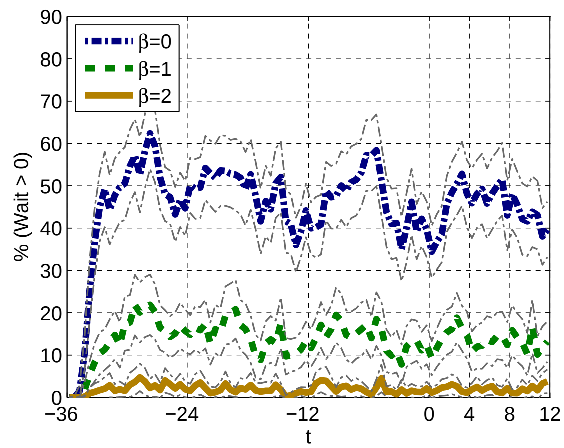

Figure 115: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

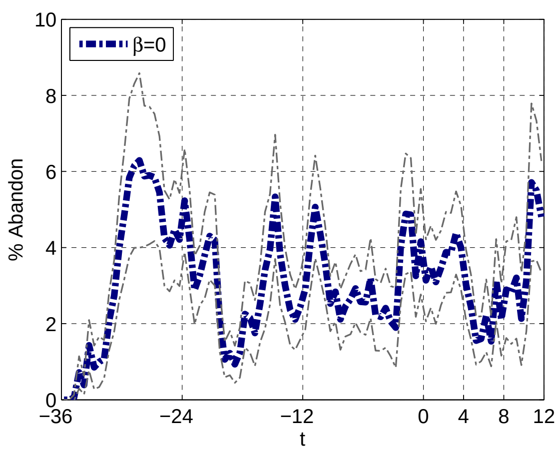

Figure 116: Average percent of arrivals abandoning: QoS $\beta = 0$.

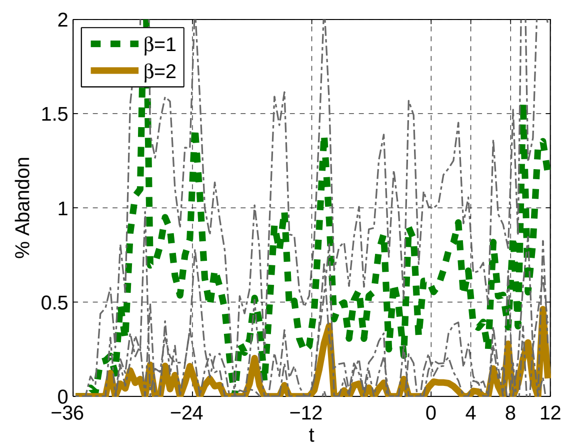

Figure 117: Average percent of arrivals abandoning: QoS $\beta = 1, 2$.

Figures 112-117: Sinusoidal arrival rate and M service time distribution. Average performance over periods of length 0.54 with associated 95% confidence intervals based on 100 replications.

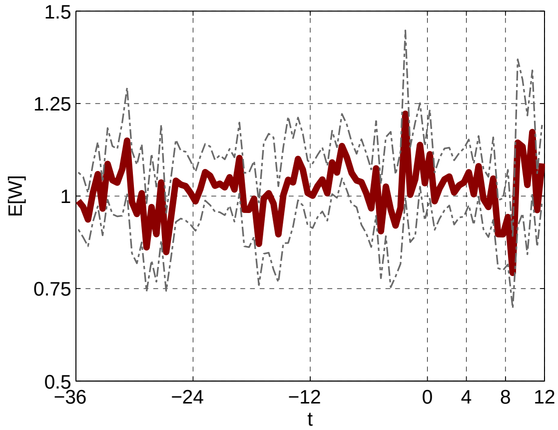

Figure 118: Average waiting time: QoS parameter $\beta = 0$.

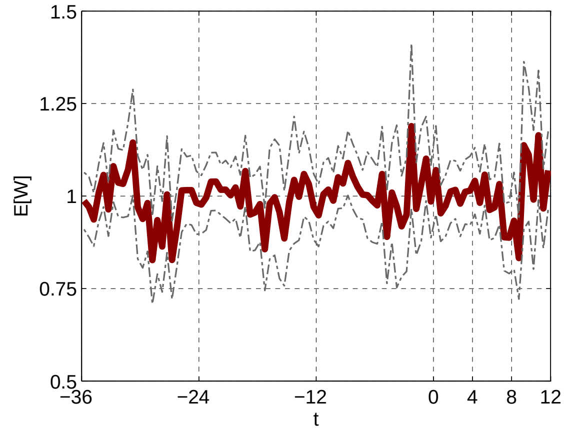

Figure 119: Average waiting time: QoS parameter $\beta = 1$.

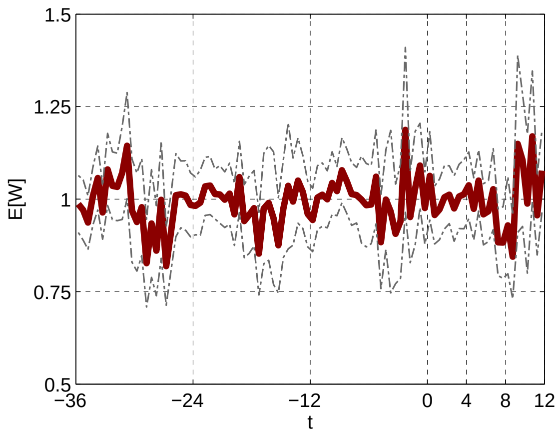

Figure 120: Average waiting time: QoS parameter $\beta = 2$.

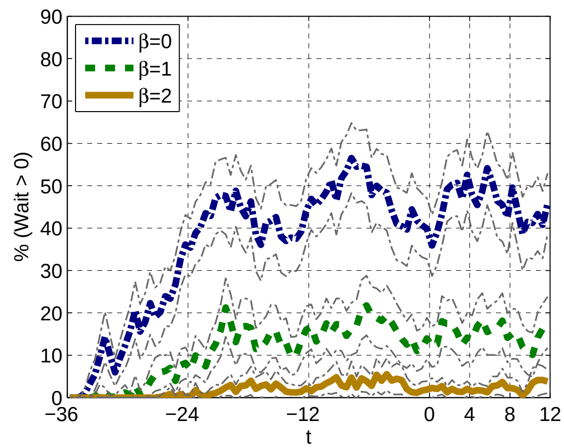

Figure 121: Average percent of arrivals delayed: QoS parameter $\beta = 0$, 1 and 2.

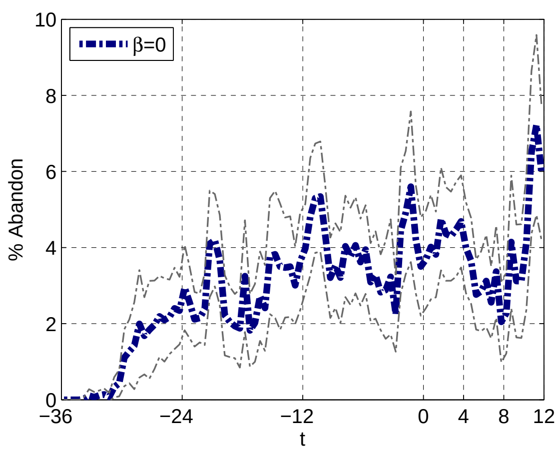

Figure 122: Average percent of arrivals abandoning: QoS parameter $\beta = 0$.

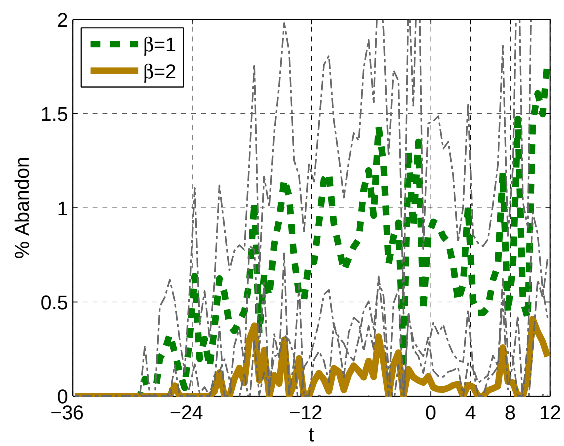

Figure 123: Average percent of arrivals abandoning: QoS parameter $\beta = 1, 2$.

Figures 118-123: Sinusoidal arrival rate and H_2 service time distribution. Average performance over periods of length 0.54 with associated 95% confidence intervals based on 100 replications.

We now present estimation results for sinusoidal arrival rate case. Table 38 provides different estimator values and Table 40 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (4.6) and (7.6)).

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	$[0,4]$	0	1.026 ± 0.014	0.886 ± 0.013	1.009 ± 0.014	1.009 ± 0.014	1.084 ± 0.018	1.006 ± 0.015	1.006 ± 0.015	1.067 ± 0.019	1.005 ± 0.015	1.005 ± 0.015
		1	1.011 ± 0.014	0.870 ± 0.011	0.990 ± 0.013	0.990 ± 0.013	1.060 ± 0.016	0.985 ± 0.013	0.985 ± 0.013	1.040 ± 0.016	0.984 ± 0.013	0.984 ± 0.013
		2	1.006 ± 0.013	0.865 ± 0.011	0.985 ± 0.012	0.985 ± 0.012	1.053 ± 0.015	0.979 ± 0.012	0.979 ± 0.012	1.033 ± 0.015	0.978 ± 0.012	0.978 ± 0.012
	Avg		1.015	0.874	0.995	0.995	1.066	0.990	0.990	1.047	0.989	0.989
	$[0,8]$	0	1.021 ± 0.011	1.017 ± 0.010	1.019 ± 0.011	1.019 ± 0.011	1.116 ± 0.012	1.028 ± 0.010	1.028 ± 0.010	1.031 ± 0.012	1.018 ± 0.011	1.018 ± 0.011
		1	1.006 ± 0.010	1.001 ± 0.009	1.005 ± 0.010	1.005 ± 0.010	1.098 ± 0.010	1.011 ± 0.009	1.011 ± 0.009	1.013 ± 0.010	1.001 ± 0.010	1.001 ± 0.010
		2	1.002 ± 0.009	0.997 ± 0.009	1.001 ± 0.009	1.001 ± 0.009	1.094 ± 0.010	1.007 ± 0.009	1.007 ± 0.009	1.009 ± 0.010	0.997 ± 0.009	0.997 ± 0.009
	Avg		1.010	1.005	1.008	1.008	1.103	1.015	1.015	1.018	1.005	1.005
H_2	$[0,4]$	0	1.035 ± 0.031	0.831 ± 0.021	0.912 ± 0.023	1.075 ± 0.032	1.104 ± 0.034	1.152 ± 0.038	1.104 ± 0.034	-1.598 ± 1.687	2.037 ± 0.215	2.037 ± 0.215
		1	1.003 ± 0.030	0.798 ± 0.015	0.874 ± 0.016	1.026 ± 0.024	1.104 ± 0.035	1.091 ± 0.025	1.091 ± 0.025	-1.599 ± 1.687	1.913 ± 0.384	1.913 ± 0.384
		2	0.999 ± 0.030	0.793 ± 0.014	0.869 ± 0.015	1.019 ± 0.022	1.104 ± 0.035	1.083 ± 0.023	1.083 ± 0.023	-1.597 ± 1.687	1.930 ± 0.458	1.930 ± 0.458
	Avg		1.012	0.808	0.885	1.040	1.104	1.109	1.093	-1.598	1.960	1.960
	$[0,8]$	0	1.025 ± 0.024	0.957 ± 0.020	0.960 ± 0.019	0.965 ± 0.022	1.077 ± 0.024	0.985 ± 0.022	0.985 ± 0.022	-1.173 ± 0.013	1.517 ± 0.056	-1.173 ± 0.013
		1	0.999 ± 0.022	0.928 ± 0.015	0.933 ± 0.016	0.943 ± 0.019	1.042 ± 0.019	0.953 ± 0.017	0.953 ± 0.017	-1.166 ± 0.013	1.432 ± 0.041	-1.166 ± 0.013
		2	0.994 ± 0.021	0.924 ± 0.014	0.929 ± 0.015	0.940 ± 0.018	1.037 ± 0.018	0.949 ± 0.016	0.949 ± 0.016	-1.165 ± 0.013	1.420 ± 0.038	-1.165 ± 0.013
	Avg		1.006	0.936	0.941	0.949	1.052	0.962	0.962	-1.168	1.456	-1.168

Table 38: SINUSOIDAL arrival rate: waiting time estimates by ten different methods (described in the beginning of §2.3) with associated 95% confidence intervals based on 100 replications.

Int	β	$\bar{W}_{L,\lambda,q,p}(t)$	# O	$\bar{W}_{L,\lambda,q,p'}(t)$
$[0,4]$	0	2.037 ± 0.21	33	1.482 ± 0.088
	1	1.913 ± 0.384	23	1.481 ± 0.079
	2	1.930 ± 0.458	22	1.505 ± 0.070
	Avg	1.960	26	1.789
$[0,8]$	0	1.517 ± 0.056	8	1.466 ± 0.048
	1	1.432 ± 0.041	0	1.432 ± 0.041
	2	1.420 ± 0.038	0	1.420 ± 0.039
	Avg	1.456	2.7	1.439

Table 39: SINUSOIDAL arrival rate with H_2 service: Waiting time estimates by $\bar{W}_{L,\lambda,q,p}(t)$ and $\bar{W}_{L,\lambda,q,p'}(t)$ with associated 95% confidence intervals. $\bar{W}_{L,\lambda,q,p'}(t)$ is the new waiting time estimate after removing the outliers ($W_{L,\lambda,q,p}(t) < 0$ or $W_{L,\lambda,q,p}(t) > 2$). The number of outliers removed in each case is given under $\#O$.

Tables 41 and 42 quantify the performance of our estimators by two performance measures.

GI	Int	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
M	$[0,4]$	0	50.9 ± 1.0	$1.35 \times 10^{-1} \pm 3.61 \times 10^{-3}$	$-4.40 \times 10^{-2} \pm 6.28 \times 10^{-3}$
		1	50.0 ± 0.9	$1.33 \times 10^{-1} \pm 3.64 \times 10^{-3}$	$-4.20 \times 10^{-2} \pm 5.94 \times 10^{-3}$
		2	49.7 ± 0.8	$1.32 \times 10^{-1} \pm 3.64 \times 10^{-3}$	$-4.14 \times 10^{-2} \pm 5.83 \times 10^{-3}$
	Avg		50.2	1.33×10^{-1}	-4.25×10^{-2}
	$[0,8]$	0	51.1 ± 0.7	$1.01 \times 10^{-2} \pm 2.52 \times 10^{-3}$	$-7.03 \times 10^{-2} \pm 1.90 \times 10^{-3}$
		1	50.2 ± 0.7	$9.94 \times 10^{-3} \pm 2.48 \times 10^{-3}$	$-6.82 \times 10^{-2} \pm 1.75 \times 10^{-3}$
		2	50.0 ± 0.6	$9.91 \times 10^{-3} \pm 2.46 \times 10^{-3}$	$-6.77 \times 10^{-2} \pm 1.70 \times 10^{-3}$
	Avg		50.5	9.98×10^{-3}	-6.87×10^{-2}
H_2	$[0,4]$	0	47.8 ± 1.4	$3.81 \times 10^{-1} \pm 1.34 \times 10^{-2}$	$-1.11 \times 10^{0} \pm 1.87 \times 10^{-1}$
		1	45.9 ± 1.0	$3.66 \times 10^{-1} \pm 1.15 \times 10^{-2}$	$-1.12 \times 10^{0} \pm 4.56 \times 10^{-1}$
		2	45.6 ± 1.0	$3.64 \times 10^{-1} \pm 1.12 \times 10^{-2}$	$-1.16 \times 10^{0} \pm 5.49 \times 10^{-1}$
	Avg		46.4	3.70×10^{-1}	-1.13×10^{0}
	$[0,8]$	0	48.1 ± 1.2	$2.84 \times 10^{-2} \pm 6.96 \times 10^{-3}$	$-7.68 \times 10^{-1} \pm 3.09 \times 10^{-2}$
		1	46.6 ± 0.9	$2.76 \times 10^{-2} \pm 6.75 \times 10^{-3}$	$-7.27 \times 10^{-1} \pm 2.44 \times 10^{-2}$
		2	46.4 ± 0.9	$2.75 \times 10^{-2} \pm 6.71 \times 10^{-3}$	$-7.20 \times 10^{-1} \pm 2.31 \times 10^{-2}$
	Avg		47.0	2.78×10^{-2}	-7.38×10^{-1}

Table 40: SINUSOIDAL arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) with associated 95% confidence intervals based on 100 replications.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
M	$[0,4]$	0	14.0	1.8	1.8	5.8	2.1	2.1	4.1	2.1	2.1	2.1
		1	14.1	2.1	2.1	4.9	2.6	2.6	3.0	2.7	2.7	2.7
		2	14.1	2.1	2.1	4.6	2.7	2.7	2.6	2.8	2.8	2.8
	Avg		14.1	2.0	2.0	5.1	2.4	2.4	3.2	2.5	2.5	2.5
	$[0,8]$	0	0.3	0.1	0.1	9.6	0.7	0.7	1.1	0.2	0.2	0.2
		1	0.5	0.1	0.1	9.3	0.5	0.5	0.7	0.5	0.5	0.5
		2	0.5	0.1	0.1	9.2	0.5	0.5	0.7	0.5	0.5	0.5
	Avg		0.4	0.1	0.1	9.3	0.6	0.6	0.8	0.4	0.4	0.4
H_2	$[0,4]$	0	20.4	12.2	4.0	6.9	11.7	6.9	263.2	100.2	100.2	44.7
		1	20.5	12.9	2.2	10.1	8.8	8.8	260.2	91.0	91.0	47.8
		2	20.6	13.0	2.0	10.5	8.4	8.4	259.7	93.1	93.1	50.6
	Avg		20.5	12.7	2.8	9.2	9.6	8.1	261.0	94.7	94.7	47.7
	$[0,8]$	0	6.8	6.5	6.0	5.1	4.1	4.1	219.8	49.2	219.8	44.1
		1	7.1	6.6	5.6	4.3	4.6	4.6	216.5	43.3	216.5	43.3
		2	7.1	6.5	5.4	4.3	4.5	4.5	215.9	42.6	215.9	42.6
	Avg		7.0	6.6	5.7	4.6	4.4	4.4	217.4	45.0	217.4	43.3

Table 41: SINUSOIDAL arrival rate: absolute difference of the waiting time estimates from the direct estimate $\bar{W}(t)$, in units of 10^{-2}.

GI	Int	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,p'}(t)$
M	$[0,4]$	0	13.6 ± 1.0	3.7 ± 0.5	3.7 ± 0.5	6.6 ± 1.1	4.7 ± 0.6	5.8 ± 1.0	4.8 ± 0.6	4.8 ± 0.6
		1	13.8 ± 0.9	3.7 ± 0.5	3.7 ± 0.5	6.2 ± 1.0	4.7 ± 0.6	5.2 ± 0.8	4.8 ± 0.6	4.8 ± 0.6
		2	13.9 ± 0.9	3.6 ± 0.5	3.6 ± 0.5	6.1 ± 1.0	4.8 ± 0.6	5.1 ± 0.8	4.9 ± 0.6	4.9 ± 0.6
	Avg		13.7 ± 0.9	3.7 ± 0.5	3.7 ± 0.5	6.3 ± 1.0	4.8 ± 0.6	5.4 ± 0.9	4.8 ± 0.6	4.8 ± 0.6
	$[0,8]$	0	2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.5 ± 0.7	2.2 ± 0.3	2.5 ± 0.4	2.2 ± 0.3	2.2 ± 0.3
		1	2.3 ± 0.3	1.5 ± 0.2	1.5 ± 0.2	9.3 ± 0.7	2.1 ± 0.3	2.4 ± 0.3	2.2 ± 0.3	2.2 ± 0.3
		2	2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.2 ± 0.7	2.1 ± 0.3	2.4 ± 0.3	2.2 ± 0.3	2.2 ± 0.3
	Avg		2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.3 ± 0.7	2.2 ± 0.3	2.4 ± 0.4	2.2 ± 0.3	2.2 ± 0.3
H_2	$[0,4]$	0	19.4 ± 2.1	13.6 ± 1.7	12.4 ± 1.9	19.1 ± 3.6	17.4 ± 2.8	397.1 ± 139.4	99.9 ± 19.2	55.8 ± 7.6
		1	19.9 ± 2.1	14.2 ± 1.8	11.8 ± 1.9	20.6 ± 4.0	16.1 ± 2.6	403.3 ± 140.7	97.3 ± 39.0	56.2 ± 7.3
		2	20.0 ± 2.1	14.4 ± 1.8	12.0 ± 1.9	20.7 ± 4.0	15.8 ± 2.5	403.5 ± 140.4	101.1 ± 47.0	55.7 ± 6.8
	Avg		19.8 ± 2.1	14.1 ± 1.8	12.0 ± 1.9	20.2 ± 3.9	16.4 ± 2.6	401.3 ± 140.2	99.4 ± 35.1	55.9 ± 7.2
	$[0,8]$	0	8.6 ± 1.1	8.0 ± 1.1	7.8 ± 1.1	9.0 ± 1.3	7.6 ± 1.1	215.8 ± 2.7	47.8 ± 3.9	44.8 ± 3.6
		1	9.0 ± 1.1	8.4 ± 1.1	7.9 ± 1.1	8.8 ± 1.3	7.8 ± 1.1	218.0 ± 2.7	43.8 ± 3.4	43.8 ± 3.4
		2	9.0 ± 1.1	8.4 ± 1.1	8.0 ± 1.1	8.7 ± 1.3	7.8 ± 1.1	218.5 ± 2.7	43.3 ± 3.3	43.3 ± 3.3
	Avg		8.9 ± 1.1	8.3 ± 1.1	7.9 ± 1.1	8.8 ± 1.3	7.7 ± 1.1	217.5 ± 2.7	44.9 ± 3.5	43.9 ± 3.5

Table 42: SINUSOIDAL arrival rate: average of the absolute relative error of the waiting time estimate from the direct estimate $\bar{W}(t)$ in each sample path with associated 95% confidence interval based on 100 replications.

3 Additional Results for the Call Center Data in Section 9

In this section, we compare the performance of the different estimators using the same call center data as in [4]. The data are for one class of customers from an American bank on 18 weekdays in May 2001. In this case, we have data for waiting times as well as arrivals and the number in the system, so that we can compare all the indirect estimators for $E[W]$ based on $\bar{L}(t)$ and the estimated arrival rate to the direct sample mean $\bar{W}(t)$ in (1).

We plot the finite averages $\bar{\lambda}(t)$ and $\bar{W}_{L,\lambda}(t)$ in (1) and (2) in Section 3.1 for 18 weekdays in May. We divide each day into 5 intervals, $[7,10]$, $[10,13]$, $[13,16]$, $[16,18]$ and $[18,22]$, so that the arrival rate is increasing in $[7,10]$, approximately stationary in $[10,13]$ and $[13,16]$, decreasing in $[16,18]$ and again decreasing in $[18,22]$ but with less steep slope.

These plots suggest that Assumption 1 is often reasonable for this data. Figures 3, 5 and 6 of [4] show that the waiting times are relatively stationary over the day, unlike the arrival rate and the number in the system. Nevertheless, we observe that the waiting times do fluctuate over time substantially for some days, especially outside of normal business hours ($[9,17]$, i.e., nine to five). Possible reasons are inappropriate time-varying staffing and the lower call volumes outside of normal business hours.

We now describe how the arrival rate approximations were done. We used an iterated least squares fit, as specified in [5], to fit a linear arrival rate function for each interval of each day. It minimizes the sum of squared deviations of the data from the model with a constraint requiring that the estimated rate function be nonnegative throughout the interval. We counted the number of arrivals in each one-minute subinterval and used that as a point in the least square fit. Similarly, a least-square fit was used to fit a quadratic arrival rate function.

We provide two sets of estimation results: for the selected 3 days in §3.2 and all 18 days in §3.3.

3.1 Plots for Arrival Rates and Time Spent in the System

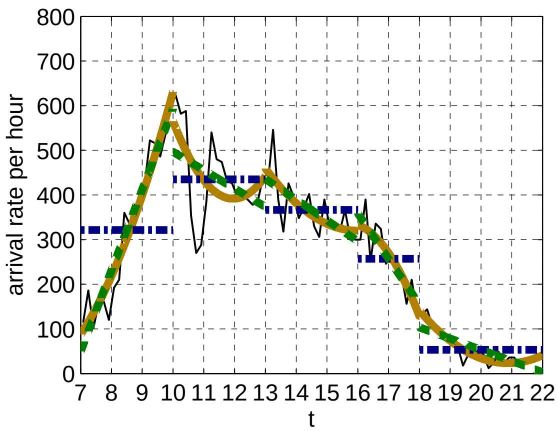

Figure 124: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 1.

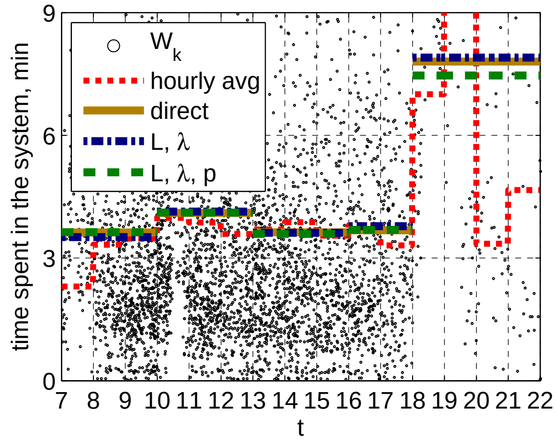

Figure 125: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 1.

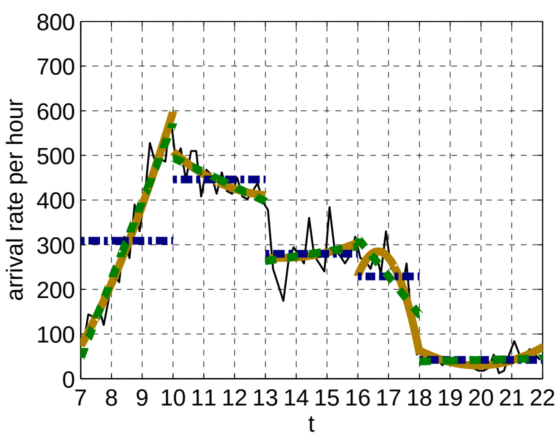

Figure 126: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 2.

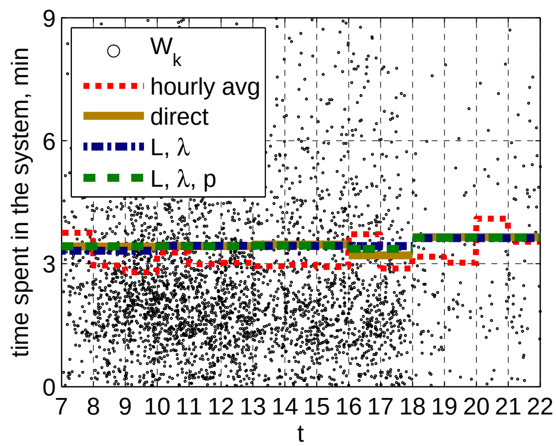

Figure 127: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 2.

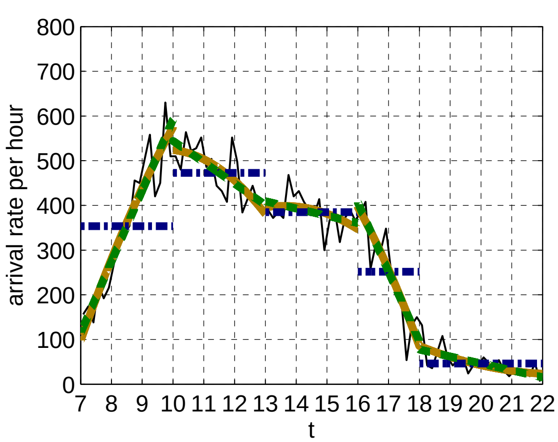

Figure 128: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 4.

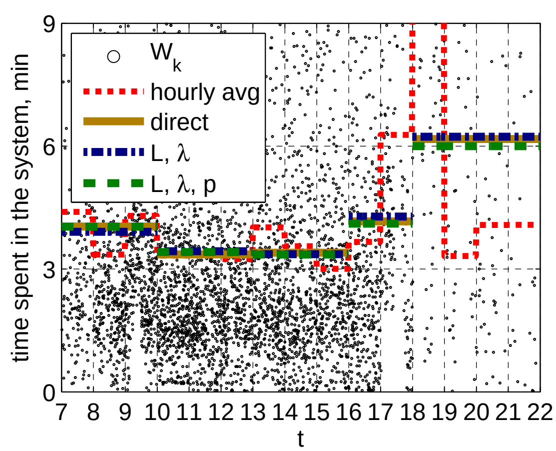

Figure 129: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 4.

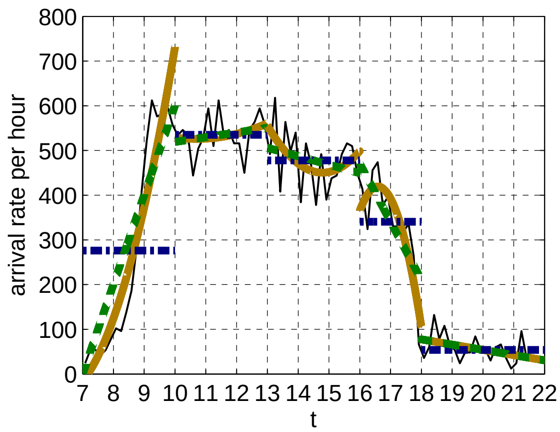

Figure 130: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, [7, 10], [10, 13], [13, 16], [16, 18] and [18, 22], on May 7.

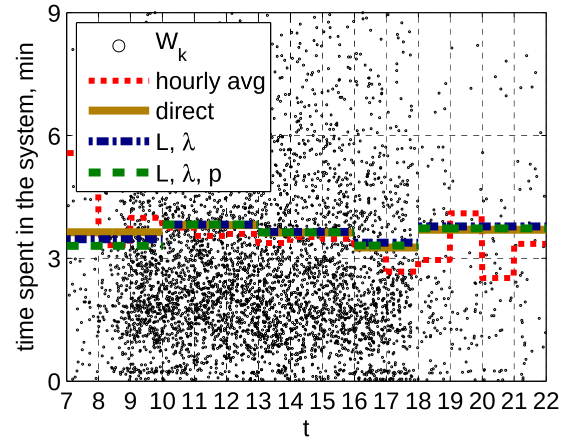

Figure 131: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in [7, 22] on May 7.

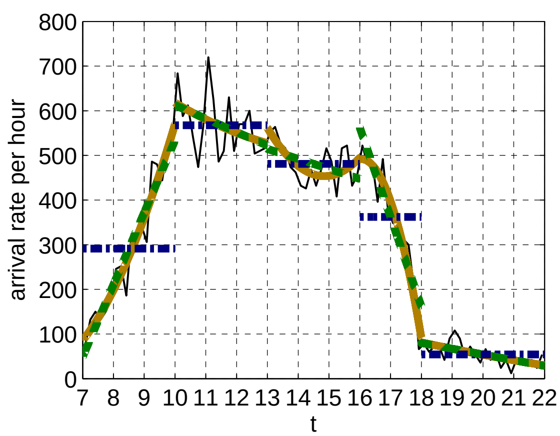

Figure 132: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, [7, 10], [10, 13], [13, 16], [16, 18] and [18, 22], on May 8.

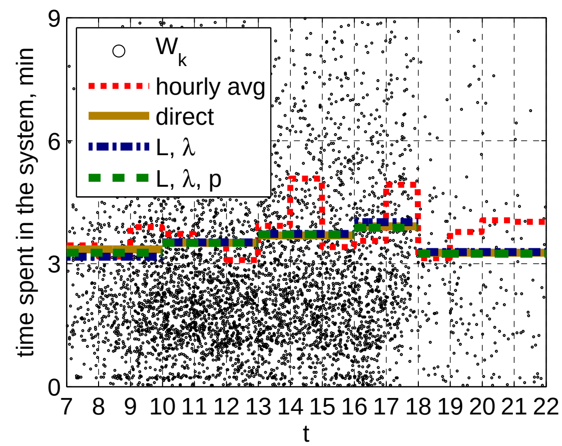

Figure 133: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in [7, 22] on May 8.

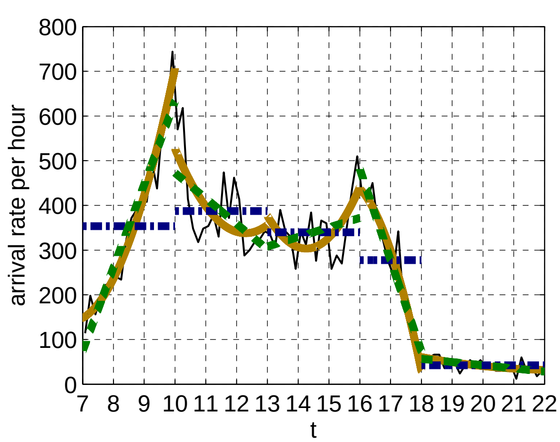

Figure 134: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, [7, 10], [10, 13], [13, 16], [16, 18] and [18, 22], on May 11.

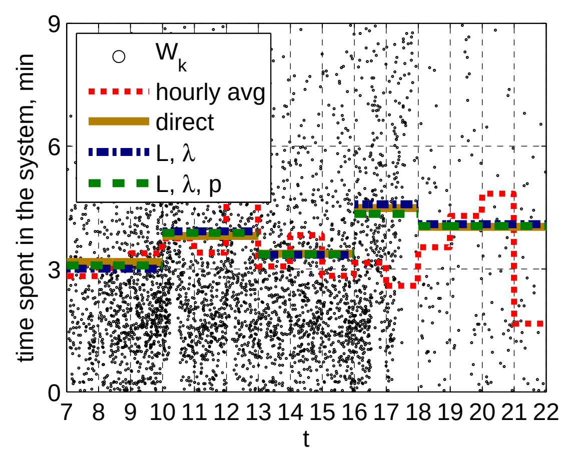

Figure 135: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in [7, 22] on May 11.

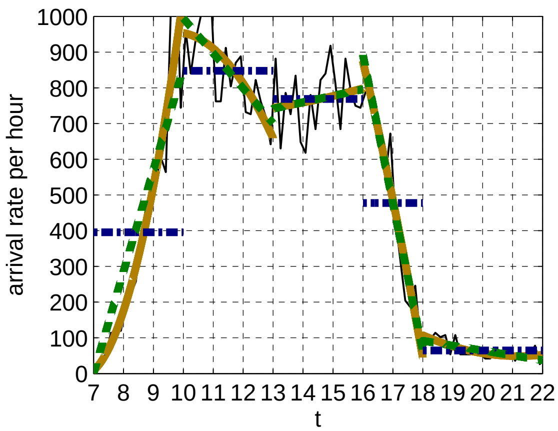

Figure 136: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 14.

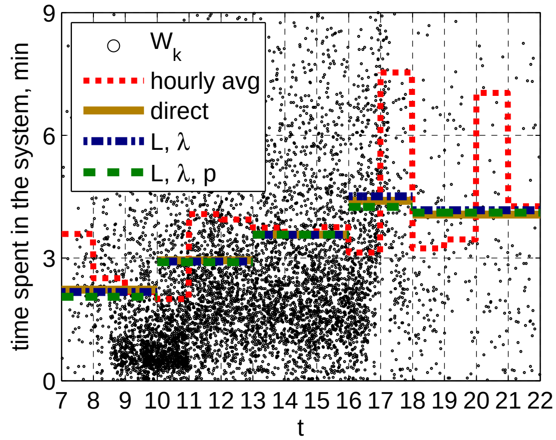

Figure 137: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 14.

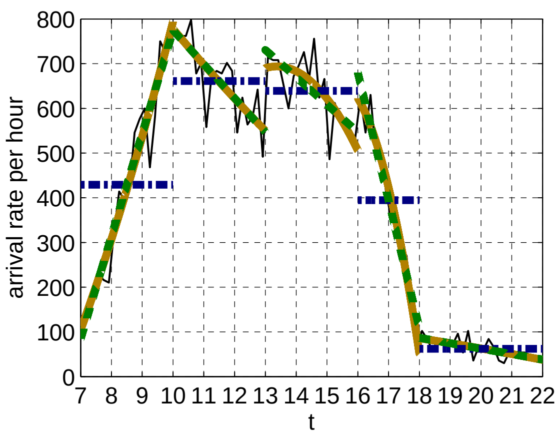

Figure 138: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 15.

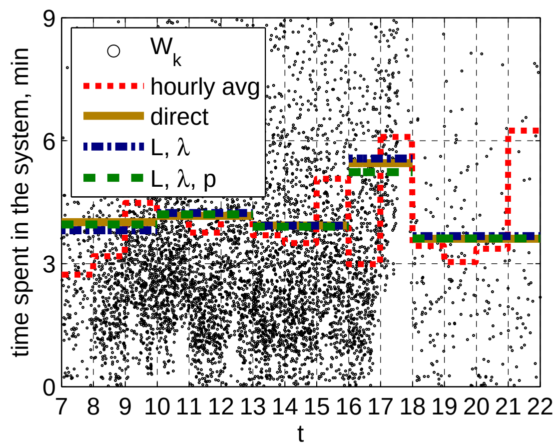

Figure 139: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 15.

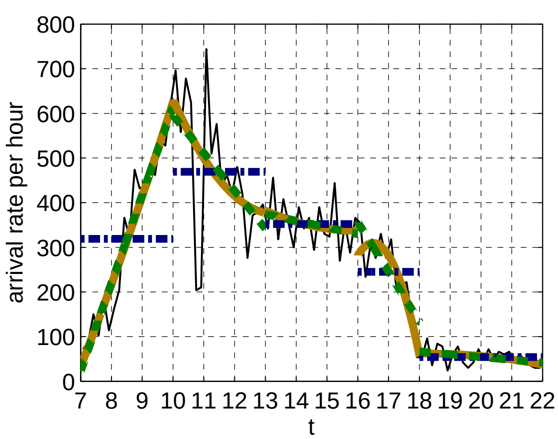

Figure 140: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 16.

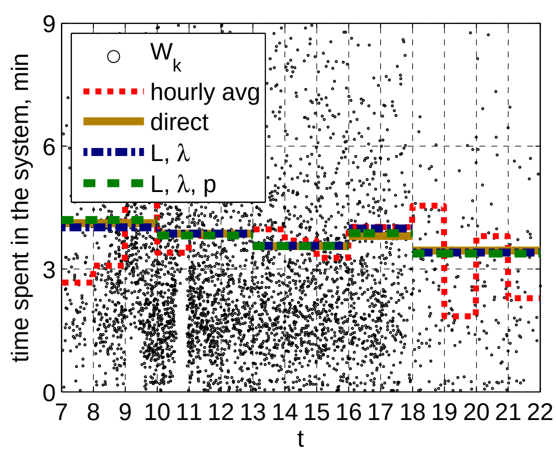

Figure 141: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 16.

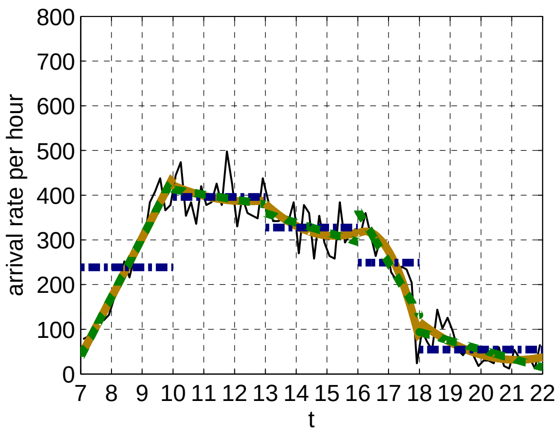

Figure 142: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 17.

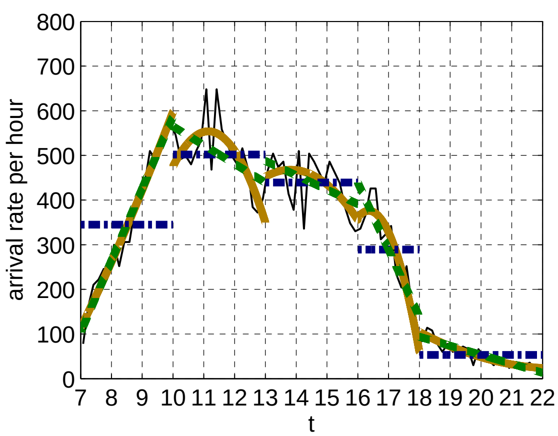

Figure 144: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 18.

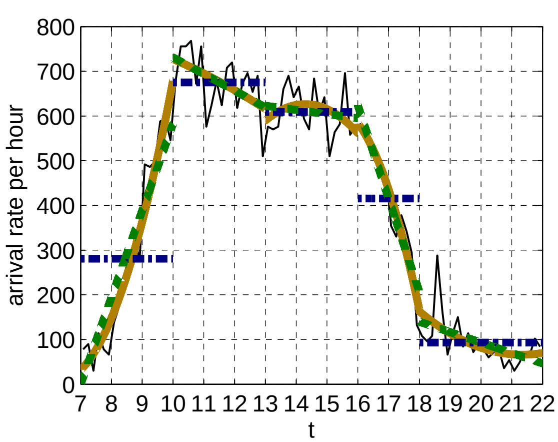

Figure 146: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 21.

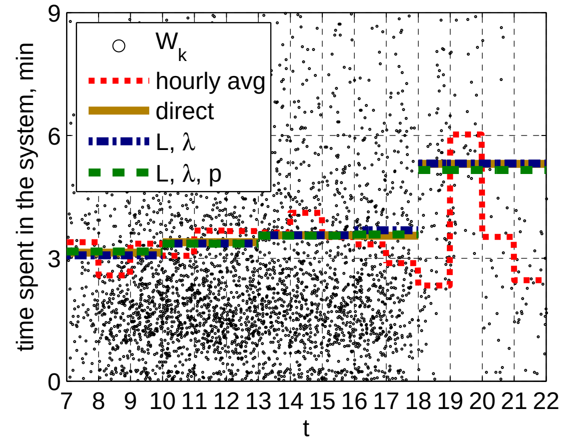

Figure 143: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 17.

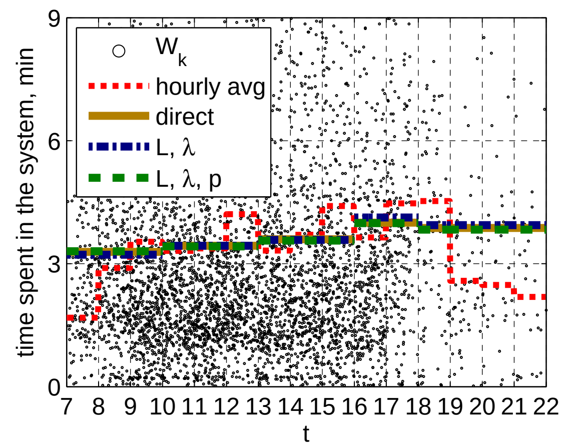

Figure 145: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 18.

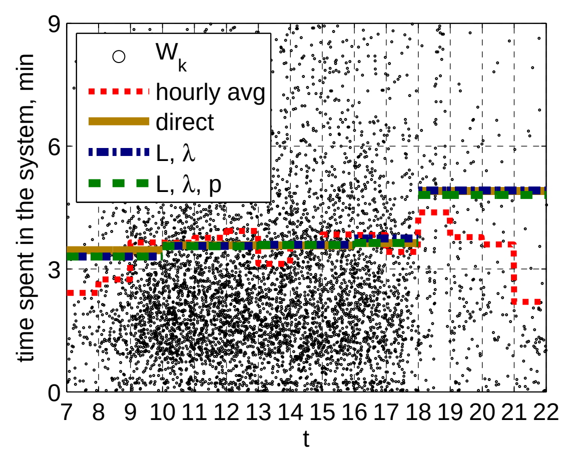

Figure 147: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 21.

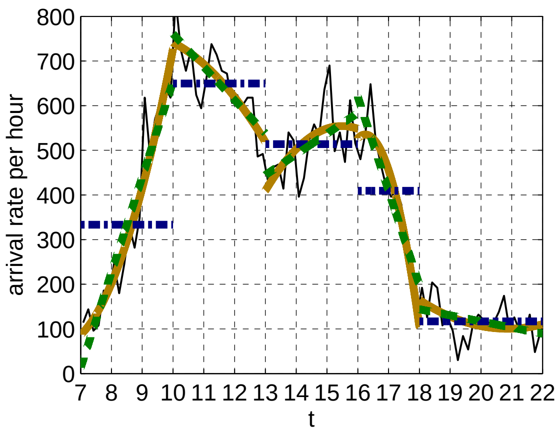

Figure 148: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 22.

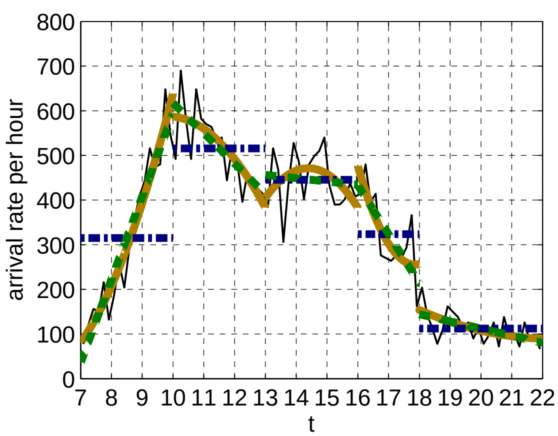

Figure 150: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 23.

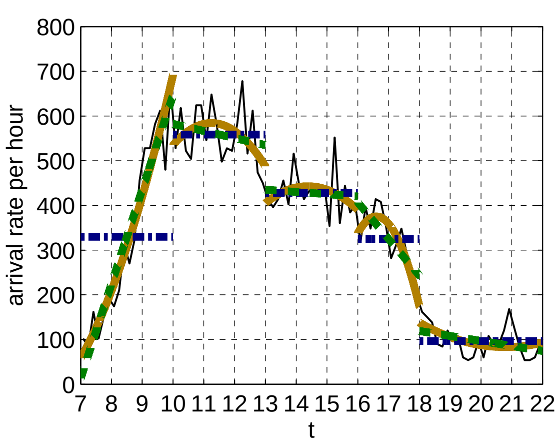

Figure 152: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 24.

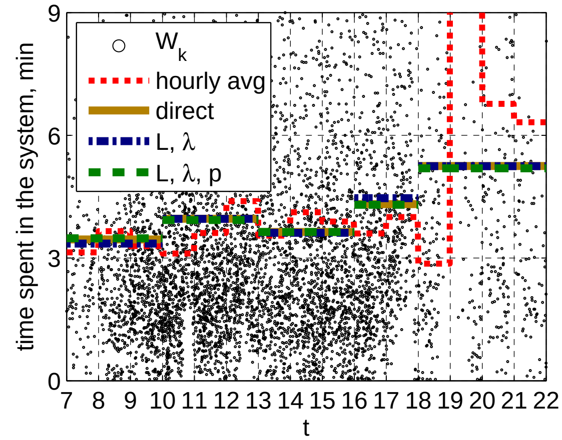

Figure 149: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 22.

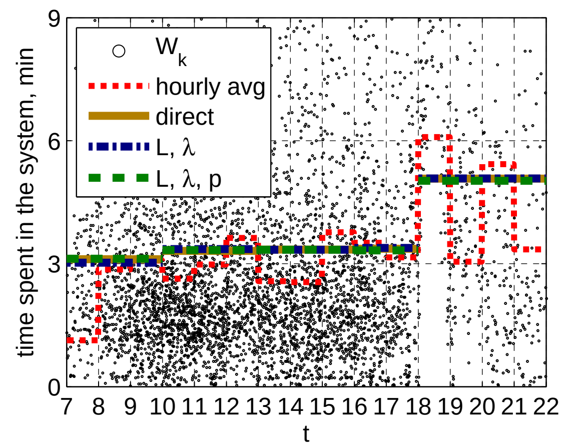

Figure 151: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 23.

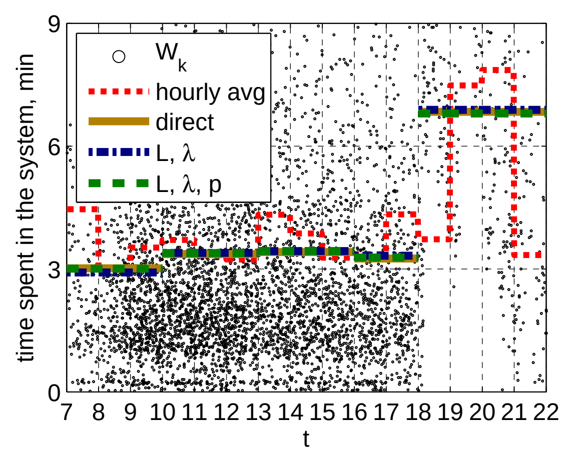

Figure 153: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 24.

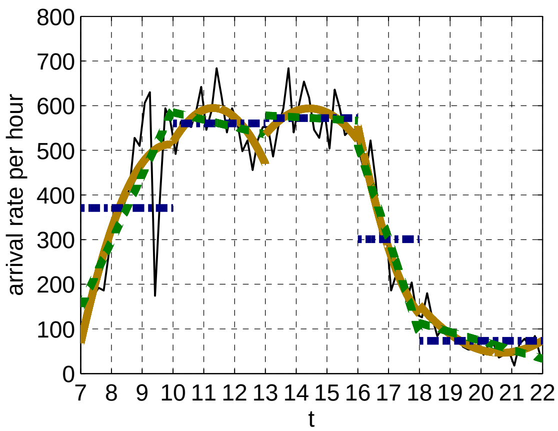

Figure 154: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 25.

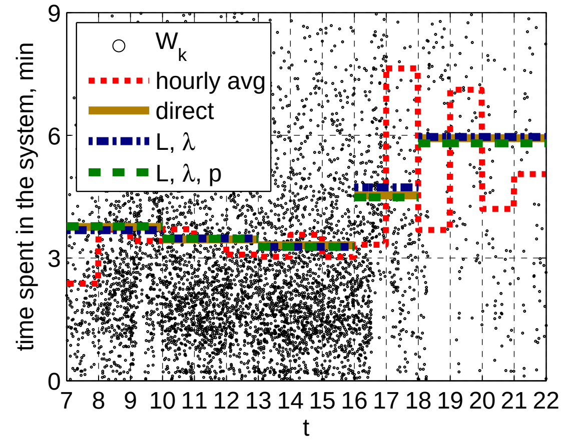

Figure 155: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 25.

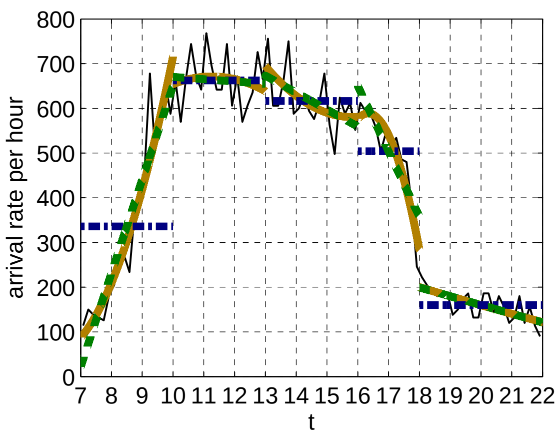

Figure 156: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 29.

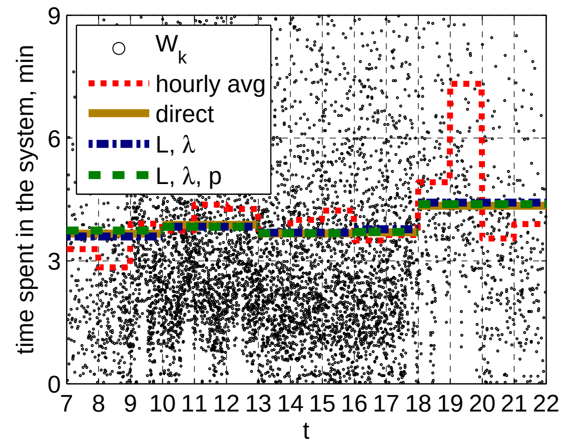

Figure 157: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 29.

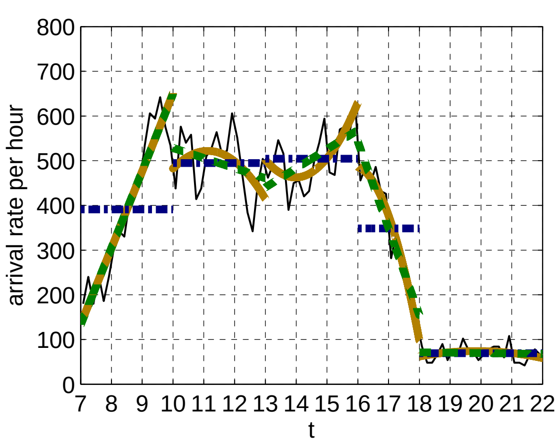

Figure 158: Arrival rate and its approximations by constant, linear and quadratic functions fitted to 5 intervals, $[7, 10]$, $[10, 13]$, $[13, 16]$, $[16, 18]$ and $[18, 22]$, on May 30.

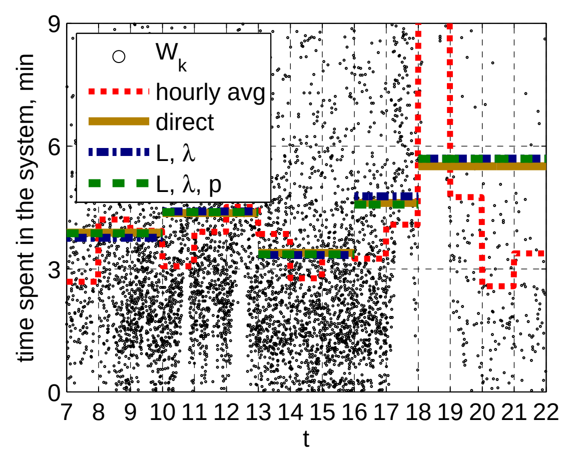

Figure 159: Scatter plot of the waiting times and its hourly averages, $\bar{W}(t)$, $\bar{W}_{L,\lambda}(t)$ and $\bar{W}_{L,\lambda,p}(t)$ of each hour in $[7, 22]$ on May 30.

3.2 Selected 3 days

		Constant	Linear		Quadratic		
Int.	*Day*	$\bar{\lambda}(t)$	a	b	a	b	c
[7, 10]	507	276.0	0.0	200.3	2050.4	−676.5	54.5
	518	345.0	103.3	161.4	−528.4	43.0	7.0
	521	280.7	0.0	194.3	2006.8	−630.7	49.8
	Avg	300.6	34.4	185.3	1176.3	−421.4	37.1
[10, 13]	507	535.0	519.1	10.6	1265.2	−138.4	6.5
	518	502.0	565.0	−42.0	−6847.3	1327.9	−59.6
	521	675.3	731.2	−37.2	782.8	18.9	−2.4
	Avg	570.8	605.1	−22.9	−1599.8	402.8	−18.5
[13, 16]	507	477.7	506.0	−18.9	7637.2	−972.0	32.9
	518	439.3	487.6	−32.2	−4044.4	653.1	−23.6
	521	609.0	623.0	−9.3	−3905.4	634.3	−22.2
	Avg	508.7	538.9	−20.1	−104.2	105.1	−4.3
[16, 18]	507	340.5	470.5	−130.0	−42905.0	5223.9	−157.5
	518	289.0	439.3	−150.3	−31042.0	3840.9	−117.4
	521	415.5	624.3	−208.8	−13647.0	1865.6	−61.0
	Avg	348.3	511.4	−163.1	−29198.0	3643.5	−112.0
[18, 22]	507	54.0	77.5	−11.7	288.8	−11.7	0.0
	518	53.3	93.9	−20.2	2062.3	−181.2	4.0
	521	93.0	139.9	−23.3	4016.0	−370.1	8.7
	Avg	66.8	103.7	−18.4	2122.4	−187.7	4.2

Table 43: Fitting constant, linear and quadratic arrival rate functions for 3 days. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Int.	*Day*	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
[7, 10]	507	3.64	3.47	3.62	3.30	3.60	0.79	3.38
	518	3.29	3.22	3.31	3.30	3.30	−2.52	3.20
	521	3.46	3.30	3.42	3.30	3.42	0.76	3.23
	Avg	3.46	3.33	3.45	3.30	3.44	−0.32	3.27
[10, 13]	507	3.81	3.82	3.81	3.83	3.83	1.89	3.80
	518	3.44	3.44	3.43	3.42	3.42	−0.34	3.39
	521	3.58	3.57	3.58	3.55	3.55	3.00	3.57
	Avg	3.61	3.61	3.60	3.60	3.60	1.52	3.59
[13, 16]	507	3.63	3.65	3.64	3.64	3.64	0.28	3.61
	518	3.60	3.58	3.57	3.56	3.56	−0.50	3.54
	521	3.58	3.59	3.60	3.59	3.59	−0.73	3.55
	Avg	3.60	3.61	3.60	3.60	3.60	−0.32	3.57
[16, 18]	507	3.26	3.38	3.25	3.31	3.31	−0.03	3.36
	518	4.00	4.13	4.03	3.99	3.98	−0.04	4.09
	521	3.63	3.75	3.62	3.64	3.63	−0.13	3.71
	Avg	3.63	3.75	3.63	3.65	3.64	−0.07	3.72
[18, 22]	507	3.70	3.78	3.73	3.73	3.73	0.77	3.77
	518	3.87	3.94	3.85	3.83	3.84	0.12	3.92
	521	4.91	4.92	4.85	4.81	4.82	0.14	4.88
	Avg	4.16	4.21	4.14	4.12	4.13	0.34	4.19

Table 44: Call center example: waiting time estimates (described in the beginning of §2.3) by different methods for the 3 days.

Int.	*Day*	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)$ $\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
$[7, 10]$	507	15.94	3.85×10^{-2}	2.89×10^{-4}
	518	18.52	2.51×10^{-2}	-8.92×10^{-5}
	521	15.45	3.67×10^{-2}	2.39×10^{-4}
	Avg	16.64	3.34×10^{-2}	1.46×10^{-4}
$[10, 13]$	507	34.09	1.26×10^{-3}	4.82×10^{-5}
	518	28.78	-4.79×10^{-3}	7.58×10^{-5}
	521	40.13	-3.28×10^{-3}	-2.15×10^{-5}
	Avg	34.33	-2.27×10^{-3}	3.42×10^{-5}
$[13, 16]$	507	29.02	-2.41×10^{-3}	3.80×10^{-5}
	518	26.20	-4.37×10^{-3}	5.24×10^{-5}
	521	36.47	-9.17×10^{-4}	5.15×10^{-5}
	Avg	30.57	-2.56×10^{-3}	4.73×10^{-5}
$[16, 18]$	507	19.19	-2.15×10^{-2}	2.60×10^{-5}
	518	19.88	-3.58×10^{-2}	3.99×10^{-5}
	521	25.94	-3.14×10^{-2}	3.94×10^{-5}
	Avg	21.67	-2.96×10^{-2}	3.51×10^{-5}
$[18, 22]$	507	3.40	-1.37×10^{-2}	4.48×10^{-18}
	518	3.50	-2.48×10^{-2}	1.99×10^{-5}
	521	7.62	-2.05×10^{-2}	3.45×10^{-5}
	Avg	4.84	-1.97×10^{-2}	1.82×10^{-5}

Table 45: Call center example: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6) for the 3 days.

Int.	*Day*	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
$[7, 10]$	507	4.76	0.62	9.18	1.09	78.20	7.14
	518	2.08	0.48	0.41	0.38	176.72	2.83
	521	4.46	1.17	4.47	0.96	77.95	6.58
	Avg	3.77	0.76	4.69	0.81	110.96	5.51
$[10, 13]$	507	0.30	0.08	0.42	0.42	50.35	0.41
	518	0.00	0.33	0.47	0.48	109.98	1.32
	521	0.44	0.15	0.77	0.77	16.29	0.36
	Avg	0.25	0.19	0.56	0.56	58.88	0.70
$[13, 16]$	507	0.51	0.44	0.27	0.27	92.36	0.34
	518	0.52	0.67	0.95	0.95	113.96	1.62
	521	0.48	0.54	0.39	0.39	120.31	0.66
	Avg	0.50	0.55	0.54	0.54	108.87	0.87
$[16, 18]$	507	3.59	0.51	1.46	1.36	100.93	2.83
	518	3.13	0.63	0.32	0.56	101.09	2.19
	521	3.19	0.17	0.14	0.05	103.62	2.23
	Avg	3.30	0.44	0.64	0.66	101.88	2.42
$[18, 22]$	507	2.18	0.76	0.82	0.78	79.21	1.89
	518	1.86	0.53	1.00	0.67	96.85	1.22
	521	0.29	1.06	1.95	1.77	97.19	0.55
	Avg	1.44	0.78	1.26	1.07	91.09	1.22

Table 46: Call center example: absolute relative error of the estimates from the direct estimate for the 3 days, in units of 10^{-2}.

3.3 All 18 days

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	321.3	49.6	181.2	576.6	−245.6	25.1
502	309.0	46.2	175.2	63.3	−120.4	17.4
504	353.7	115.0	159.1	−1819.7	354.3	−11.5
507	276.0	0.0	200.3	2050.4	−676.5	54.5
508	291.3	48.8	161.7	649.3	−250.2	24.2
511	353.7	74.2	186.3	2282.5	−648.8	49.1
514	395.3	0.0	285.0	3937.7	−1199.1	91.1
515	429.0	84.2	229.9	−556.5	−0.4	13.5
516	319.0	22.6	197.6	−879.5	83.2	6.7
517	238.7	38.4	133.5	−814.6	114.1	1.1
518	345.0	103.3	161.4	−528.4	43.0	7.0
521	280.7	0.0	194.3	2006.8	−630.7	49.8
522	333.3	12.8	213.7	2027.4	−620.9	49.1
523	315.0	36.0	186.0	855.7	−318.5	29.7
524	329.7	12.7	211.3	704.4	−304.8	30.4
525	370.7	148.4	148.2	−4730.9	1061.7	−53.7
529	336.0	21.6	209.6	1662.3	−529.3	43.5
530	391.0	132.6	172.3	−983.7	151.0	1.3
Avg	332.7	52.6	189.3	361.3	−207.7	23.8

Table 47: $[7, 10]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.62	3.51	3.63	3.63	3.63	3.83	3.39
502	3.42	3.32	3.42	3.43	3.42	51.85	3.51
504	4.04	3.91	4.05	4.03	4.02	−1.05	3.85
507	3.64	3.47	3.62	3.30	3.60	0.79	3.38
508	3.35	3.17	3.31	3.27	3.26	2.61	3.08
511	3.17	3.01	3.14	3.09	3.09	0.73	2.95
514	2.23	2.17	2.22	2.05	2.22	0.35	2.14
515	4.01	3.82	4.00	3.96	3.95	−3.18	3.80
516	4.12	4.02	4.14	4.20	4.18	−1.75	3.98
517	3.14	3.07	3.17	3.17	3.16	−1.15	3.05
518	3.29	3.22	3.31	3.30	3.30	−2.52	3.20
521	3.46	3.30	3.42	3.30	3.42	0.76	3.23
522	3.46	3.35	3.47	3.48	3.47	0.89	3.28
523	3.11	3.03	3.09	3.12	3.12	2.01	2.95
524	3.02	2.91	3.02	3.01	3.00	2.76	2.82
525	3.76	3.68	3.80	3.78	3.77	−0.41	3.62
529	3.67	3.60	3.69	3.74	3.73	1.20	3.51
530	3.89	3.76	3.86	3.87	3.87	−1.97	3.71
Avg	3.47	3.35	3.46	3.43	3.46	3.10	3.30

Table 48: $[7, 10]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
501	18.80	3.30×10^{-2}	5.67×10^{-4}
502	17.07	3.13×10^{-2}	-1.84×10^{-3}
504	23.05	2.93×10^{-2}	7.15×10^{-5}
507	15.94	3.85×10^{-2}	2.89×10^{-4}
508	15.40	2.93×10^{-2}	3.70×10^{-4}
511	17.73	2.64×10^{-2}	1.64×10^{-4}
514	14.29	2.41×10^{-2}	9.60×10^{-5}
515	27.29	3.41×10^{-2}	-2.10×10^{-4}
516	21.35	4.15×10^{-2}	-8.06×10^{-5}
517	12.23	2.87×10^{-2}	-9.20×10^{-6}
518	18.52	2.51×10^{-2}	-8.92×10^{-5}
521	15.45	3.67×10^{-2}	2.39×10^{-4}
522	18.62	3.58×10^{-2}	2.36×10^{-4}
523	15.88	2.98×10^{-2}	3.08×10^{-4}
524	16.01	3.11×10^{-2}	3.99×10^{-4}
525	22.75	2.45×10^{-2}	1.19×10^{-4}
529	20.15	3.74×10^{-2}	2.99×10^{-4}
530	24.52	2.76×10^{-2}	-1.28×10^{-5}
Avg	18.61	3.13×10^{-2}	5.07×10^{-5}

Table 49: $[7, 10]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.11	0.10	0.31	0.08	5.69	6.57
502	3.00	0.07	0.25	0.04	1417.20	2.79
504	3.21	0.35	0.19	0.37	125.99	4.74
507	4.76	0.62	9.18	1.09	78.20	7.14
508	5.27	1.04	2.31	2.49	22.16	7.87
511	5.11	0.81	2.46	2.60	77.06	6.76
514	2.56	0.42	7.79	0.21	84.07	3.92
515	4.92	0.27	1.44	1.68	179.32	5.38
516	2.48	0.47	1.93	1.56	142.53	3.40
517	1.98	1.03	1.00	0.83	136.68	2.90
518	2.08	0.48	0.41	0.38	176.72	2.83
521	4.46	1.17	4.47	0.96	77.95	6.58
522	3.00	0.30	0.75	0.47	74.14	5.09
523	2.66	0.49	0.43	0.24	35.41	5.10
524	3.42	0.00	0.21	0.41	8.57	6.42
525	2.06	1.11	0.47	0.35	111.02	3.71
529	1.85	0.69	2.13	1.83	67.24	4.23
530	3.19	0.80	0.36	0.52	150.58	4.44
	3.28	0.57	2.01	0.89	165.03	4.99

Table 50: $[7, 10]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	*a*	*b*	*a*	*b*	*c*
501	435.0	496.7	−41.1	6909.8	−1090.9	45.6
502	446.0	494.8	−32.5	2023.9	−243.1	9.2
504	472.7	544.3	−47.8	−767.8	265.3	−13.6
507	535.0	519.1	10.6	1265.2	−138.4	6.5
508	567.3	612.6	−30.2	1379.6	−111.6	3.5
511	387.3	472.8	−57.0	5778.6	−885.3	36.0
514	847.7	995.0	−98.2	−1643.3	535.0	−27.5
515	661.0	775.0	−76.0	1804.0	−123.1	2.0
516	469.0	596.0	−84.7	4515.1	−622.0	23.4
517	396.0	412.8	−11.2	1322.0	−150.6	6.1
518	502.0	565.0	−42.0	−6847.3	1327.9	−59.6
521	675.3	731.2	−37.2	782.8	18.9	−2.4
522	649.7	761.1	−74.3	−236.7	230.2	−13.2
523	515.7	617.6	−68.0	−1383.2	400.8	−20.4
524	558.7	582.4	−15.8	−3423.6	712.5	−31.7
525	560.3	585.1	−16.5	−5033.4	995.1	−44.0
529	662.7	670.5	−5.2	−704.7	244.4	−10.9
530	494.7	528.6	−22.6	−3310.9	688.5	−30.9
Avg	546.4	608.9	−41.6	135.0	114.1	−6.8

Table 51: $[10, 13]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	4.10	4.14	4.12	4.11	4.11	0.33	4.09
502	3.43	3.44	3.43	3.43	3.43	0.91	3.41
504	3.36	3.43	3.37	3.41	3.41	−4.10	3.32
507	3.81	3.82	3.81	3.83	3.83	1.89	3.80
508	3.51	3.52	3.52	3.51	3.51	1.63	3.50
511	3.81	3.92	3.81	3.88	3.88	0.33	3.88
514	2.94	2.92	2.93	2.90	2.90	−2.74	2.85
515	4.17	4.24	4.16	4.20	4.20	1.72	4.21
516	3.86	3.86	3.86	3.82	3.82	0.50	3.82
517	3.39	3.36	3.37	3.36	3.36	1.19	3.34
518	3.44	3.44	3.43	3.42	3.42	−0.34	3.39
521	3.58	3.57	3.58	3.55	3.55	3.00	3.57
522	3.96	3.96	3.95	3.93	3.93	−453.96	4.69
523	3.31	3.36	3.34	3.33	3.33	−2.08	3.28
524	3.36	3.40	3.37	3.39	3.39	−0.78	3.35
525	3.45	3.48	3.45	3.47	3.47	−0.53	3.43
529	3.88	3.83	3.86	3.83	3.83	−7.37	3.69
530	4.37	4.41	4.41	4.39	4.39	−0.92	4.32
Avg	3.65	3.67	3.65	3.65	3.65	−25.63	3.66

Table 52: $[10, 13]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
501	29.99	-6.52×10^{-3}	7.83×10^{-5}
502	25.57	-4.18×10^{-3}	3.51×10^{-5}
504	27.04	-5.78×10^{-3}	2.04×10^{-4}
507	34.09	1.26×10^{-3}	4.82×10^{-5}
508	33.26	-3.12×10^{-3}	1.97×10^{-5}
511	25.31	-9.61×10^{-3}	6.60×10^{-5}
514	41.20	-5.63×10^{-3}	1.34×10^{-4}
515	46.67	-8.12×10^{-3}	1.24×10^{-5}
516	30.19	-1.16×10^{-2}	5.20×10^{-5}
517	22.19	-1.59×10^{-3}	3.37×10^{-5}
518	28.78	-4.79×10^{-3}	7.58×10^{-5}
521	40.13	-3.28×10^{-3}	-2.15×10^{-5}
522	42.85	-7.54×10^{-3}	-2.64×10^{-3}
523	28.87	-7.38×10^{-3}	1.45×10^{-4}
524	31.61	-1.60×10^{-3}	8.05×10^{-5}
525	32.48	-1.71×10^{-3}	7.83×10^{-5}
529	42.35	-5.02×10^{-4}	2.23×10^{-4}
530	36.33	-3.35×10^{-3}	1.36×10^{-4}
Avg	33.27	-4.73×10^{-3}	-6.87×10^{-5}

Table 53: $[10, 13]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	0.84	0.38	0.19	0.19	91.90	0.39
502	0.22	0.16	0.20	0.20	73.55	0.52
504	2.22	0.42	1.63	1.63	222.16	1.00
507	0.30	0.08	0.42	0.42	50.35	0.41
508	0.17	0.11	0.14	0.14	53.63	0.32
511	2.94	0.11	1.97	1.95	91.26	1.79
514	0.89	0.42	1.44	1.44	193.05	3.26
515	1.69	0.06	0.87	0.86	58.74	1.17
516	0.04	0.04	1.18	1.20	87.18	1.02
517	0.87	0.79	1.03	1.03	64.85	1.53
518	0.00	0.33	0.47	0.48	109.98	1.32
521	0.44	0.15	0.77	0.77	16.29	0.36
522	0.07	0.32	0.81	0.82	11564.00	18.45
523	1.35	0.69	0.61	0.60	162.86	0.95
524	0.92	0.26	0.76	0.76	123.09	0.53
525	0.70	0.14	0.53	0.53	115.40	0.68
529	1.17	0.62	1.22	1.22	289.86	4.80
530	0.92	0.99	0.58	0.58	121.14	0.96
	0.87	0.34	0.82	0.82	749.41	2.19

Table 54: $[10, 13]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	366.3	435.9	−46.4	4302.3	−498.1	15.6
502	279.7	263.3	10.9	1263.1	−147.1	5.4
504	385.0	409.4	−16.2	−1288.5	248.0	−9.1
507	477.7	506.0	−18.9	7637.2	−972.0	32.9
508	481.0	514.5	−22.3	7389.5	−933.8	31.4
511	340.0	308.0	21.3	9592.7	−1302.3	45.6
514	768.7	741.5	18.1	850.2	−29.5	1.6
515	639.0	730.9	−61.3	−4164.3	726.6	−27.2
516	352.0	374.4	−14.9	1645.6	−164.0	5.1
517	327.7	359.7	−21.4	3591.2	−430.2	14.1
518	439.3	487.6	−32.2	−4044.4	653.1	−23.6
521	609.0	623.0	−9.3	−3905.4	634.3	−22.2
522	514.0	444.7	46.2	−5042.2	722.6	−23.3
523	445.3	456.4	−7.4	−6622.3	985.7	−34.2
524	427.7	435.2	−5.0	−3699.7	576.4	−20.0
525	572.3	578.2	−3.9	−5354.6	824.4	−28.6
529	616.3	673.2	−37.9	4193.1	−456.9	14.4
530	504.3	440.3	42.7	8463.6	−1144.7	40.9
Avg	474.7	487.9	−8.8	822.6	−39.3	1.1

Table 55: $[13, 16]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.63	3.62	3.58	3.59	3.59	0.37	3.59
502	3.47	3.43	3.43	3.44	3.44	0.91	3.41
504	3.40	3.36	3.37	3.36	3.36	−1.38	3.32
507	3.63	3.65	3.64	3.64	3.64	0.28	3.61
508	3.69	3.72	3.70	3.71	3.71	0.29	3.69
511	3.38	3.35	3.37	3.36	3.36	0.15	3.32
514	3.57	3.57	3.58	3.57	3.57	3.38	3.56
515	3.93	3.93	3.96	3.90	3.90	−0.80	3.88
516	3.56	3.57	3.54	3.56	3.56	0.89	3.54
517	3.56	3.56	3.57	3.55	3.55	0.39	3.54
518	3.60	3.58	3.57	3.56	3.56	−0.50	3.54
521	3.58	3.59	3.60	3.59	3.59	−0.73	3.55
522	3.64	3.62	3.62	3.64	3.64	−0.46	3.58
523	3.34	3.34	3.34	3.34	3.34	−0.28	3.31
524	3.42	3.44	3.41	3.44	3.44	−0.51	3.41
525	3.31	3.27	3.29	3.27	3.27	−0.45	3.24
529	3.67	3.69	3.67	3.68	3.68	0.64	3.66
530	3.39	3.34	3.36	3.36	3.36	0.25	3.31
Avg	3.54	3.54	3.53	3.53	3.53	0.13	3.50

Table 56: $[13, 16]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
501	22.08	-7.64×10^{-3}	3.10×10^{-5}
502	16.01	2.23×10^{-3}	3.32×10^{-5}
504	21.58	-2.36×10^{-3}	5.91×10^{-5}
507	29.02	-2.41×10^{-3}	3.80×10^{-5}
508	29.86	-2.88×10^{-3}	3.91×10^{-5}
511	18.99	3.50×10^{-3}	3.60×10^{-5}
514	45.72	1.40×10^{-3}	1.43×10^{-5}
515	41.84	-6.28×10^{-3}	7.19×10^{-5}
516	20.93	-2.52×10^{-3}	2.54×10^{-5}
517	19.47	-3.87×10^{-3}	3.28×10^{-5}
518	26.20	-4.37×10^{-3}	5.24×10^{-5}
521	36.47	-9.17×10^{-4}	5.15×10^{-5}
522	30.99	5.42×10^{-3}	4.13×10^{-5}
523	24.81	-9.20×10^{-4}	3.98×10^{-5}
524	24.52	-6.72×10^{-4}	4.46×10^{-5}
525	31.22	-3.72×10^{-4}	3.97×10^{-5}
529	37.91	-3.79×10^{-3}	3.04×10^{-5}
530	28.07	4.71×10^{-3}	3.63×10^{-5}
Avg	28.09	-1.21×10^{-3}	3.98×10^{-5}

Table 57: $[13, 16]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	0.36	1.26	1.11	1.12	89.87	1.11
502	0.98	1.09	0.76	0.76	73.89	1.68
504	0.99	0.65	1.22	1.22	140.61	2.29
507	0.51	0.44	0.27	0.27	92.36	0.34
508	0.98	0.29	0.70	0.69	92.02	0.12
511	0.73	0.24	0.38	0.38	95.66	1.56
514	0.07	0.15	0.07	0.07	5.42	0.25
515	0.10	0.68	0.72	0.73	120.28	1.44
516	0.24	0.52	0.01	0.01	75.10	0.38
517	0.02	0.08	0.40	0.40	89.05	0.79
518	0.52	0.67	0.95	0.95	113.96	1.62
521	0.48	0.54	0.39	0.39	120.31	0.66
522	0.54	0.47	0.01	0.00	112.71	1.51
523	0.05	0.20	0.14	0.14	108.49	0.99
524	0.72	0.06	0.66	0.66	114.90	0.31
525	1.17	0.76	1.20	1.20	113.47	2.12
529	0.50	0.06	0.12	0.12	82.58	0.23
530	1.57	0.99	1.10	1.11	92.78	2.39
	0.58	0.51	0.57	0.57	96.30	1.10

Table 58: $[13, 16]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	257.0	359.0	−102.0	−9346.5	1233.3	−39.3
502	229.0	314.6	−85.6	−34631.0	4191.7	−125.8
504	251.5	407.0	−155.5	−1699.7	385.6	−15.9
507	340.5	470.5	−130.0	−42905.0	5223.9	−157.5
508	362.0	562.4	−200.4	−26816.0	3401.9	−106.0
511	277.5	481.5	−204.0	−15081.0	2013.5	−65.2
514	477.5	892.0	−414.5	−402.4	519.1	−27.5
515	394.5	680.4	−285.9	−19460.0	2625.0	−85.6
516	245.0	356.5	−111.5	−30032.0	3677.9	−111.5
517	249.0	366.3	−117.3	−20275.0	2534.9	−78.0
518	289.0	439.3	−150.3	−31042.0	3840.9	−117.4
521	415.5	624.3	−208.8	−13647.0	1865.6	−61.0
522	409.5	620.3	−210.8	−36324.0	4537.8	−139.7
523	323.5	433.4	−109.9	20863.0	−2309.1	64.7
524	325.0	406.8	−81.8	−28059.0	3425.1	−103.1
525	301.0	512.3	−210.8	22246.0	−2373.3	63.6
529	504.0	650.3	−146.3	−28964.0	3617.4	−110.7
530	348.5	543.4	−194.4	−21076.0	2718.3	−85.7
Avg	333.3	506.7	−173.3	−17592.0	2285.0	−72.3

Table 59: $[16, 18]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.68	3.78	3.74	3.69	3.68	−0.12	3.75
502	3.20	3.43	3.29	3.36	3.36	−0.03	3.40
504	4.16	4.28	4.17	4.11	4.09	−0.81	4.20
507	3.26	3.38	3.25	3.31	3.31	−0.03	3.36
508	3.91	4.02	3.89	3.88	3.87	−0.06	3.98
511	4.49	4.58	4.47	4.35	4.32	−0.10	4.53
514	4.40	4.51	4.33	4.25	4.22	−517.80	6.61
515	5.45	5.57	5.38	5.23	5.19	−0.13	5.49
516	3.81	3.99	3.85	3.87	3.87	−0.04	3.95
517	3.55	3.69	3.53	3.58	3.58	−0.05	3.66
518	4.00	4.13	4.03	3.99	3.98	−0.04	4.09
521	3.63	3.75	3.62	3.64	3.63	−0.13	3.71
522	4.32	4.46	4.34	4.30	4.29	−0.06	4.42
523	3.36	3.39	3.34	3.32	3.32	0.06	3.36
524	3.25	3.33	3.25	3.28	3.28	−0.04	3.30
525	4.53	4.73	4.51	4.48	4.47	0.07	4.68
529	3.70	3.78	3.70	3.71	3.71	−0.07	3.74
530	4.61	4.77	4.58	4.57	4.56	−0.09	4.72
Avg	3.96	4.09	3.96	3.94	3.93	−28.86	4.16

Table 60: $[16, 18]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
501	16.19	-2.50×10^{-2}	3.75×10^{-5}
502	13.09	-2.14×10^{-2}	2.65×10^{-5}
504	17.95	-4.41×10^{-2}	1.17×10^{-4}
507	19.19	-2.15×10^{-2}	2.60×10^{-5}
508	24.23	-3.70×10^{-2}	3.96×10^{-5}
511	21.18	-5.61×10^{-2}	5.65×10^{-5}
514	35.89	-6.53×10^{-2}	-3.01×10^{-2}
515	36.60	-6.72×10^{-2}	8.47×10^{-5}
516	16.29	-3.03×10^{-2}	3.65×10^{-5}
517	15.30	-2.89×10^{-2}	3.25×10^{-5}
518	19.88	-3.58×10^{-2}	3.99×10^{-5}
521	25.94	-3.14×10^{-2}	3.94×10^{-5}
522	30.45	-3.83×10^{-2}	4.74×10^{-5}
523	18.25	-1.92×10^{-2}	2.18×10^{-5}
524	18.01	-1.40×10^{-2}	2.52×10^{-5}
525	23.71	-5.51×10^{-2}	3.89×10^{-5}
529	31.71	-1.83×10^{-2}	3.38×10^{-5}
530	27.73	-4.43×10^{-2}	5.75×10^{-5}
Avg	22.87	-3.63×10^{-2}	-1.63×10^{-3}

Table 61: $[16, 18]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	2.72	1.72	0.27	0.15	103.23	1.80
502	7.12	2.68	4.93	4.83	100.80	6.33
504	2.87	0.21	1.31	1.67	119.44	0.91
507	3.59	0.51	1.46	1.36	100.93	2.83
508	2.69	0.43	0.85	1.11	101.58	1.76
511	2.01	0.38	3.15	3.72	102.15	0.89
514	2.57	1.51	3.37	4.12	11876.00	50.33
515	2.10	1.40	3.97	4.77	102.38	0.72
516	4.75	1.12	1.76	1.58	100.97	3.84
517	3.79	0.59	0.95	0.79	101.45	2.94
518	3.13	0.63	0.32	0.56	101.09	2.19
521	3.19	0.17	0.14	0.05	103.62	2.23
522	3.36	0.59	0.32	0.59	101.32	2.34
523	0.66	0.74	1.20	1.27	98.25	0.01
524	2.19	0.01	0.80	0.76	101.34	1.45
525	4.23	0.45	1.11	1.51	98.43	3.30
529	2.06	0.14	0.27	0.20	102.02	1.21
530	3.45	0.71	0.90	1.13	101.95	2.31
	3.14	0.78	1.50	1.68	756.52	4.85

Table 62: $[16, 18]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	*a*	*b*	*a*	*b*	*c*
501	53.3	103.5	−25.1	6220.8	−593.5	14.2
502	42.0	38.5	1.8	3714.3	−370.2	9.3
504	46.3	77.6	−15.5	1554.9	−135.7	3.0
507	54.0	77.5	−11.7	288.8	−11.7	0.0
508	54.3	80.0	−12.9	489.3	−30.7	0.4
511	42.5	56.8	−7.1	771.7	−66.0	1.5
514	64.5	91.9	−13.7	2631.6	−243.8	5.8
515	62.3	86.7	−12.1	181.7	0.2	−0.3
516	54.3	66.6	−6.2	−509.4	62.7	−1.7
517	54.5	94.1	−19.8	3825.0	−358.4	8.5
518	53.3	93.9	−20.2	2062.3	−181.2	4.0
521	93.0	139.9	−23.3	4016.0	−370.1	8.7
522	116.8	144.1	−13.7	3499.8	−325.7	7.8
523	112.3	144.4	−16.1	1846.9	−157.9	3.5
524	96.8	119.1	−11.2	3118.1	−291.9	7.0
525	73.3	113.3	−20.0	6524.9	−627.2	15.2
529	160.3	199.5	−19.6	661.8	−30.6	0.3
530	69.5	71.3	−0.9	−1285.6	136.9	−3.4
Avg	72.4	99.9	−13.7	2200.7	−199.7	4.6

Table 63: $[18, 22]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	7.80	7.90	7.57	7.46	7.41	0.08	7.79
502	3.65	3.63	3.65	3.63	3.63	0.05	3.60
504	6.17	6.23	6.06	6.00	6.02	0.22	6.17
507	3.70	3.78	3.73	3.73	3.73	0.77	3.77
508	3.27	3.29	3.26	3.24	3.24	0.41	3.27
511	4.03	4.10	3.95	4.05	4.05	0.27	4.07
514	4.06	4.17	4.12	4.11	4.11	0.12	4.14
515	3.61	3.67	3.50	3.62	3.63	1.27	3.67
516	3.45	3.40	3.38	3.38	3.38	−0.47	3.37
517	5.31	5.32	5.32	5.16	5.15	0.09	5.27
518	3.87	3.94	3.85	3.83	3.84	0.12	3.92
521	4.91	4.92	4.85	4.81	4.82	0.14	4.88
522	5.25	5.24	5.26	5.19	5.19	0.21	5.20
523	5.07	5.08	5.02	5.02	5.02	0.37	5.04
524	6.84	6.89	6.83	6.80	6.79	0.26	6.80
525	5.93	5.96	5.88	5.81	5.80	0.08	5.90
529	4.35	4.42	4.36	4.39	4.38	1.18	4.41
530	5.51	5.70	5.49	5.69	5.69	−0.38	5.63
Avg	4.82	4.87	4.78	4.77	4.77	0.27	4.83

Table 64: $[18, 22]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (4.6)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (7.6)
501	7.01	-6.21×10^{-2}	9.39×10^{-5}
502	2.54	2.52×10^{-3}	2.22×10^{-5}
504	4.80	-3.47×10^{-2}	4.89×10^{-5}
507	3.40	-1.37×10^{-2}	$4.48 \times 10^{-1}8$
508	2.97	-1.30×10^{-2}	6.16×10^{-6}
511	2.90	-1.15×10^{-2}	2.09×10^{-5}
514	4.48	-1.47×10^{-2}	2.52×10^{-5}
515	3.81	-1.19×10^{-2}	-1.28×10^{-5}
516	3.07	-6.43×10^{-3}	2.77×10^{-5}
517	4.83	-3.22×10^{-2}	4.14×10^{-5}
518	3.50	-2.48×10^{-2}	1.99×10^{-5}
521	7.62	-2.05×10^{-2}	3.45×10^{-5}
522	10.20	-1.02×10^{-2}	4.05×10^{-5}
523	9.51	-1.21×10^{-2}	3.23×10^{-5}
524	11.10	-1.32×10^{-2}	7.02×10^{-5}
525	7.27	-2.71×10^{-2}	5.48×10^{-5}
529	11.82	-9.03×10^{-3}	4.92×10^{-6}
530	6.60	-1.24×10^{-3}	5.90×10^{-5}
Avg	5.97	-1.76×10^{-2}	3.28×10^{-5}

Table 65: $[18, 22]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (4.6) and (7.6).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	1.24	3.03	4.37	5.05	98.95	0.17
502	0.68	0.09	0.43	0.43	98.62	1.35
504	0.93	1.80	2.86	2.57	96.41	0.07
507	2.18	0.76	0.82	0.78	79.21	1.89
508	0.51	0.42	0.76	0.80	87.33	0.14
511	1.58	2.00	0.44	0.42	93.34	0.94
514	2.65	1.46	1.18	1.14	96.96	1.92
515	1.84	3.06	0.27	0.64	64.89	1.81
516	1.46	1.92	2.09	2.10	113.62	2.26
517	0.16	0.16	2.87	3.06	98.27	0.75
518	1.86	0.53	1.00	0.67	96.85	1.22
521	0.29	1.06	1.95	1.77	97.19	0.55
522	0.09	0.12	1.09	1.11	95.97	0.99
523	0.13	0.98	1.05	1.08	92.75	0.66
524	0.71	0.07	0.58	0.62	96.21	0.48
525	0.49	0.88	2.10	2.24	98.63	0.57
529	1.81	0.22	0.91	0.89	72.93	1.44
530	3.43	0.29	3.30	3.30	106.98	2.25
	1.23	1.05	1.56	1.59	93.62	1.08

Table 66: $[18, 22]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

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